

# TWISTED AND CONICAL KÄHLER-RICCI SOLITON ON FANO MANIFOLD

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**ABSTRACT.** In this paper, we consider the twisted Kähler-Ricci soliton, and show that the existence of twisted Kähler-Ricci soliton with semi-positive twisting form is closely related to the properness of some energy functionals. We also consider the conical Kähler-Ricci soliton, and obtain some existence results. In particular, under some assumptions on the divisor and  $\alpha$ -invariant, we get the properness of the modified log K-energy and the existence of conical Kähler-Ricci soliton with suitable cone angle.

## 1. INTRODUCTION

Let  $(M, J)$  be a compact Fano manifold. A Kähler metric  $\omega \in 2\pi c_1(M)$  is called a Kähler-Ricci soliton if there exists a holomorphic vector field  $X$  over  $M$  such that

$$\text{Ric}(\omega) = \omega + L_X \omega.$$

Kähler-Ricci solitons can be considered as a natural extension of the Kähler-Einstein metrics, which have been studied by Cao [4], Cao-Tian-Zhu [5], Hamilton [13], Tian [22], Tian-Zhu [27], Zhu [33], etc. Specially, in [5], the authors shown that the existence of Kähler-Ricci solitons is closely related to the properness of the modified Ding-functional or Mabuchi  $K$ -energy. Following the works of Aubin [1] and Yau [31], we study the continuity method used in [5]. Given a Kähler metric  $\omega_0 \in 2\pi c_1(M)$  and holomorphic vector field  $X$ , the approach is to find  $\omega_t$  solving the following equation,

$$(1.1) \quad \text{Ric}(\omega_t) = t\omega_t + (1-t)\omega_0 + L_X \omega_t$$

for all  $t \in [0, 1]$ . In this paper, we are interested in the supremum of  $t$  for which we can solve the equation above.

Let  $\text{Aut}(M)$  be the connected component containing the identity holomorphism transformation and  $\eta(M)$  be its Lie algebra consisting of all holomorphic vector fields on  $M$ . According to [10], there exists a semidirect decomposition of  $\text{Aut}(M)$ , such that

$$\text{Aut}(M) = \dot{\text{Aut}}(M) \ltimes R_u,$$

where  $R_u$  is the unipotent radical of  $\text{Aut}(M)$  and  $\dot{\text{Aut}}(M) \subset \text{Aut}(M)$  is the reductive subgroup as a complexification of a subgroup  $K$  of  $\text{Aut}(M)$ , where  $K$  is the maximal compact subgroup of  $\text{Aut}(M)$  containing the one-parameter transformations subgroup of  $\text{Aut}(M)$  generating by  $\text{Im } X$ . Obviously, the Lie subalgebra  $\dot{\eta}(M)$  of  $\dot{\text{Aut}}(M)$  is reductive. More precisely,  $\dot{\eta}(M)$  is the complexification of a real compact Lie algebra of  $K$ . In particular,  $X \in \dot{\eta}(M)$ . Now we assume  $\omega_0 \in 2\pi c_1(M)$

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is a smooth Kähler metric invariant under the action of  $\Phi_{\text{Im } X}$ , where  $\Phi_{\text{Im } X}$  is the one-parameter transformations subgroup of  $\text{Aut}(M)$  generating by  $\text{Im } X$ , i.e.

$$L_{\text{Im } X} \omega_0 = 0.$$

Let  $\mathcal{H}(M, \omega_0) = \{\varphi \in L_{loc}^1(M) \mid \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \text{ in the sense of current}\}$ . As in [5], we define the following function subspace of  $\mathcal{H}(M, \omega_0)$ :

$$\mathcal{H}_X(M, \omega_0) = \{\varphi \in \mathcal{H}(M, \omega_0) \cap C^\infty(M) \mid \text{Im}(X)\varphi = 0\}.$$

We define  $\mathcal{K}_X^0(\omega_0)$  to be the space of smooth semipositive  $(1, 1)$ -forms cohomology to  $\omega_0$ , i.e.

$$\mathcal{K}_X^0(\omega_0) = \{\omega \in [\omega_0] \mid \omega \text{ is smooth and } \omega \geq 0, L_{\text{Im } X} \omega = 0\}.$$

Furthermore,  $\mathcal{K}_X(\omega_0)$  is a subspace of Kähler metrics defined as follow:

$$\mathcal{K}_X(\omega_0) = \{\omega \in [\omega_0] \mid \omega \text{ is a Kähler metric and } L_{\text{Im } X}(\omega) = 0\}.$$

**Definition 1.1.** We define the following invariant with respect to  $X$ ,

$$(1.2) \quad R(X) = \sup\{\beta \mid \exists \omega \in \mathcal{K}_X(\omega_0), \text{ such that } \text{Ric}(\omega) - L_X \omega \geq \beta \omega\}.$$

*Remark 1.1.* According to [33], there exists  $\omega'_0 \in \mathcal{K}_X(\omega_0)$ , such that

$$\text{Ric}(\omega'_0) - L_X \omega'_0 = \omega_0 \geq c' \omega'_0 > 0,$$

so we get  $R(X) > 0$ . Furthermore, by taking integration over  $M$  on both sides of  $\text{Ric}(\omega) - L_X \omega \geq \beta \omega$ , we conclude that  $\beta \leq 1$ , therefore  $0 < R(X) \leq 1$ . If  $X \equiv 0$ ,  $R(0)$  is just the invariant defined in [12].

Note that  $\omega$  is a closed form and  $X$  is holomorphic, we have that  $\bar{\partial}(i_X \omega) = 0$ . According to the Hodge decomposition theorem and the property of Fano manifold, we can find a smooth real-valued function  $\theta_X(\omega)$  such that for  $\omega \in \mathcal{K}_X(\omega_0)$ ,

$$(1.3) \quad i_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X(\omega)$$

and  $\theta_X(\omega)$  satisfies the normalization

$$\int_M e^{\theta_X(\omega)} \omega^n = \int_M \omega_0^n.$$

We will take notation that  $\theta_X = \theta_X(\omega_0)$  in the whole paper without special instruction. By direct computation, we get that  $\theta_X(\omega_\varphi) = \theta_X + X(\varphi)$ .

**Definition 1.2.** For any  $(1, 1)$ -form  $\eta \in (1 - \beta)\mathcal{K}_X^0(\omega_0)$ , we say a Kähler metric  $\omega \in \mathcal{K}_X(\omega_0)$  is a twisted Kähler–Ricci soliton with respect to  $\eta$  if it satisfies

$$(1.4) \quad \text{Ric}(\omega) = \beta \omega + \eta + L_X \omega.$$

*Remark 1.2.* It is easy to see that finding the twisted Kähler–Ricci soliton as (1.4) is equivalent to solving the following Monge–Ampère equation:

$$(1.5) \quad \frac{\omega_\varphi^n}{\omega_0^n} = e^{h_{\omega_0} - \beta \varphi - \theta_X - X(\varphi)},$$

where  $h_\omega$  is the Ricci potential defined by

$$(1.6) \quad \text{Ric}(\omega) - \beta \omega - \eta = \sqrt{-1} \partial \bar{\partial} h_\omega$$

normalized such that  $\int_M e^{h_\omega} \omega^n = \int_M \omega^n$ . And we just consider the case when  $\beta$  is nonnegative, since the equation (1.5) is solvable according to the celebrated work of Aubin [1] and Yau [31] on the other case.

In this paper, we will follow Tian's argument in [22] to show that the existence of twisted Kähler–Ricci soliton with respect to  $\eta$  is closely related to the properness of the twisted  $K$ -energy  $\tilde{\mu}_{\omega_0, \eta}$  (see the definition in section 1). Following the discussion of Tian–Zhu [25] and Phong–Song–Strum–Weinkove [19], we deduce a linear Moser–Trudinger type inequality. In fact, we get the first main theorem.

**Theorem 1.1.** *Let  $(M, \omega_0)$  be a compact Kähler manifold,  $L_{\text{Im } X} \omega_0 = 0$  and  $\eta$  is a real closed semipositive  $(1, 1)$ -form in  $(1 - \beta)\mathcal{K}_X^0(\omega_0)$  with  $\beta > 0$ , where  $X$  is a holomorphic vector field on  $M$ . Suppose the twisted  $K$ -energy  $\tilde{\mu}_{\omega_0, \eta}$  is proper. Then there is a twisted Kähler–Ricci soliton  $\omega \in [\omega_0]$  with respect to  $\eta$ , i.e.*

$$(1.7) \quad \text{Ric}(\omega) = \beta\omega + \eta + L_X\omega.$$

On the other hand, assuming that the twisted form  $\eta$  is strictly positive at a point, if there exists a twisted Kähler–Ricci soliton  $\omega_{TKS} \in \mathcal{K}_X(\omega_0)$  with respect to  $\eta$ , then  $\tilde{\mu}_{\omega_0, \eta}$  must be proper. Furthermore, there exist two positive constants  $C_1$  and  $C_2$  depending only on  $\beta, \eta, X$  and the geometry of  $(M, \omega_{TKS})$ , such that

$$(1.8) \quad \tilde{\mu}_{\omega_0, \eta}(\varphi) \geq C_1 \tilde{J}_{\omega_0}(\varphi) - C_2$$

for all  $\varphi \in \mathcal{H}_X(M, \omega_0)$ .

*Remark 1.3.*

- (1) A functional  $G$  is  $J$ -proper on the function space  $\mathcal{H}$  if there exists an increasing function  $f : \mathbb{R} \rightarrow [c, +\infty)$  satisfying  $\lim_{t \rightarrow +\infty} f(t) = +\infty$ , such that

$$G(\phi) \geq f(J(\phi))$$

for any  $\phi \in \mathcal{H}$ .

- (2) When  $\eta$  is strictly positive at one point, following the argument in [3], we can get the uniqueness of twisted Kähler–Ricci soliton.

If  $\beta \in (0, R(X))$ , by the definition of  $R(X)$ , there exists a Kähler metric  $\tilde{\omega} \in \mathcal{K}_X(\omega_0)$  such that

$$\text{Ric}(\tilde{\omega}) - \beta\tilde{\omega} - L_X\tilde{\omega} > 0.$$

Let  $\eta = \text{Ric}(\tilde{\omega}) - \beta\tilde{\omega} - L_X\tilde{\omega} \in (1 - \beta)\mathcal{K}_X(\omega_0)$ . The equation (1.7) can be solved in  $\mathcal{K}_X(\omega_0)$ . By Theorem 1.1, we obtain that the twisted  $K$ -energy  $\tilde{\mu}_{\omega_0, \eta}$  is proper, in fact, it satisfies the Moser–Trudinger type inequality (1.8). On other hand, by the definition of  $\tilde{\mu}_{\omega_0, \eta}$ , it is easy to see that the properness of the twisted  $K$ -energy  $\tilde{\mu}_{\omega_0, \eta}$  is independent on the choice of the twisting form  $\eta \in (1 - \beta)\mathcal{K}_X(\omega_0)$ , which implies that  $\tilde{\mu}_{\omega_0, (1-\beta)\omega_0}$  is also proper. Then the equation (1.1) can be solved at  $t = \beta$ . Furthermore, since  $\omega_0$  is strictly positive, we have the following corollary,

**Corollary 1.2.** *Let  $(M, \omega_0)$  be a Kähler manifold with  $\omega_0 \in 2\pi c_1(M)$ , and  $0 < \beta < 1$ . The following conditions are equivalent:*

- (1) we can solve the equation (1.1),
- (2) there exists a Kähler metric  $\omega \in \mathcal{K}_X(\omega_0)$  such that  $\text{Ric}(\omega) - L_X(\omega) > \beta\omega$ ,
- (3) for any Kähler metric  $\omega \in \mathcal{K}_X(\omega_0)$ ,  $\tilde{\mu}_\omega + (1 - \beta)(\tilde{I}_\omega - \tilde{J}_\omega)$  is proper.

Let  $D = \{s = 0\} \in |L|$  be a smooth divisor, and particularly, in this paper we consider  $L$  is a holomorphic line bundle such that  $c_1(L) = \lambda c(M)$ , for some  $\lambda \in \mathbb{Q}^+$ . A smooth conical Kähler metric on  $M$  with angle  $2\pi\beta$  ( $0 < \beta < 1$ ) along

$D$  is a closed positive  $(1, 1)$ -current which is a smooth Kähler metric in  $M \setminus D$  and asymptotically equivalent to the model conical metric

$$\sqrt{-1} \sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j + \sqrt{-1} |z^n|^{2\beta-2} dz^n \wedge d\bar{z}^n,$$

where  $(z^1, \dots, z^n)$  are local holomorphic coordinates such that  $D = \{z^n = 0\}$ . As in [8], we give the definition of conical Kähler-Ricci soliton with respect to the holomorphic vector field  $X$ , which has been studied on the toric manifold by [8] and [29].

**Definition 1.3.** A conical Kähler metric  $\omega \in 2\pi c_1(M)$  is called a conical Kähler-Ricci soliton with respect to  $X$  if  $\omega$  satisfies:

- (1) The metric potential of  $\omega$  is Hölder continuous with respect to  $\omega_0$  on  $M$ ,
- (2)  $Ric(\omega) = \gamma(\lambda, \nu)\omega + \nu[D] + L_X\omega$  globally on  $M$  in the sense of current, where  $[D]$  is the current of integration along  $D$ ,
- (3)  $Ric(\omega) = \gamma(\lambda, \nu)\omega + L_X\omega$  on  $M \setminus D$  in the classical sense,

where  $\gamma(\lambda, \nu) = 1 - \lambda\nu$ .

*Remark 1.4.* While dealing with  $L_X\omega$  as a current, we mean that for any smooth  $(n-1, n-1)$ -form  $\zeta$ ,

$$\int_M L_X\omega \wedge \zeta = - \int_M \omega \wedge L_X\zeta.$$

We will show that the existence of conical Kähler-Ricci soliton is also closely related to the properness of the log modified Mabuchi  $K$ -energy  $\tilde{\mu}_{\omega_0, \nu D}$  and the log Ding functional  $\tilde{F}_{\omega_0, \nu D}$  which will be defined in section 5.

**Theorem 1.3.** Assume that  $X(\log |s|_H^2)$  is bounded, where  $s$  is the defined section of  $D$  and  $H$  is a Hermitian metric on the line bundle  $L$ . If  $\tilde{\mu}_{\omega_0, \nu D}$  or  $\tilde{F}_{\omega_0, \nu D}$  is proper on the function space  $\mathcal{H}_X(M, \omega_0)$ , where  $0 < \nu < 1$ , then there exists a conical Kähler-Ricci soliton  $\omega_\nu$  with angle  $2\pi(1 - \nu)$  along  $D$ , i.e.  $\omega_\nu$  satisfies:

$$(1.9) \quad Ric(\omega_\nu) = \gamma(\lambda, \nu)\omega_\nu + \nu[D] + L_X\omega_\nu.$$

Under some assumptions on the divisor and  $\alpha$ -invariant, modifying Berman's work [2], we obtain the existence of conical Kähler-Ricci soliton for suitable cone angles, i.e. Theorem 6.2. when  $R(X) = 1$ , we prove that the supremum of the cone angle of conical Kähler-Ricci soliton must be  $2\pi$ , i.e. we get the following theorem.

**Theorem 1.4.** Assume that  $R(X) = 1$ ,  $D \in |L|$ ,  $|X(\log |s|_H^2)| < C < +\infty$  and

- (1)  $\tilde{C} < \lambda$ ,
- (2)  $\min\{\alpha(\omega_0), \lambda\alpha(L|_D)\} > \max\{\frac{\tilde{C}(1-\lambda)}{(1-\tilde{C})}, 0\}$ ,

where  $\tilde{C}$  is the positive constant  $C_2 < 1$  in Proposition 2.1, and  $\alpha(\omega_0)$  and  $\alpha(L|_D)$  are the alpha invariants defined by Tian. For any  $\beta \in (\max\{\frac{1-\lambda}{1-\tilde{C}}, 0\}, 1)$ , there exists a conical metric  $\omega_\beta$  with the cone angle  $2\pi(1 - \frac{1-\beta}{\lambda})$  such that

$$Ric(\omega_\beta) = \beta\omega_\beta + \frac{1-\beta}{\lambda}[D] + L_X\omega_\beta.$$

Furthermore,  $\omega_\beta$  is a Gromov–Hausdorff limit of smooth twisted Kähler–Ricci solitons.

*Remark 1.5.* Our argument in Theorem 1.4 can also be applied to the conical Kähler–Einstein case, i.e.  $X \equiv 0$ . When  $L$  is just  $-\lambda K_M$  for  $\lambda \geq 1$ , according to Bermann [2] and Li-Sun [16], the log Mabuchi  $K$ -energy for small cone angle is proper. If  $R(0)$  the greatest lower bound of Ricci tensor defined in [21] is 1, then we can get that the log Mabuchi  $K$ -energy is proper for any cone angle in  $(0, 2\pi)$ . This result has been proved by Chi Li in [15], but our argument is different to that in [15].

We will organize the paper as follow. In section 2, we will introduce the twisted Kähler–Ricci soliton and related functionals, i.e. twisted Mabuchi  $K$ -energy and Ding–functional. In section 3, we will discuss the existence of twisted Kähler–Ricci solitons. Then, we give a necessary condition for the existence of twisted Kähler–Ricci solitons, i.e. a version of Moser–Trudinger inequality, which will be proved in section 4. In section 5, we will prove the existence of conical Kähler–Ricci solitons under properness assumption of the log modified Mabuchi  $K$ -energy or log modified Ding–functional. In the last section, we find some condition under which we can get the properness, and we consider the limit behavior of a sequence of twisted Kähler–Ricci solitons in the sense of Gromov–Hausdorff distance.

## 2. SOME TWISTED FUNCTIONALS

In this section,  $\eta$  will be a fix  $(1, 1)$ -form in  $(1 - \beta)\mathcal{K}_X^0(\omega_0)$  ( $0 < \beta < 1$ ). Firstly, let us recall the modified Aubin–Yau functional  $\tilde{J}_{\omega_0}$  and  $\tilde{I}_{\omega_0}$  defined on  $\mathcal{H}_X(M, \omega_0)$  in [26]:

$$\begin{aligned}\tilde{J}_{\omega_0}(\varphi) &= \frac{1}{V} \int_0^1 \int_M \dot{\varphi}_t (e^{\theta_X} \omega_0^n - e^{\theta_X(\omega_{\varphi_s})} \omega_{\varphi_s}^n) \wedge dt, \\ \tilde{I}_{\omega_0}(\varphi) &= \frac{1}{V} \int_M \varphi (e^{\theta_X} \omega_0^n - e^{\theta_X(\omega_\varphi)} \omega_\varphi^n),\end{aligned}$$

where  $\{\varphi_t\}$  ( $0 \leq t \leq 1$ ) is a smooth path in  $\mathcal{H}_X(M, \omega_0)$  connecting 0 and  $\varphi$ , and  $\theta_X(\omega)$  is defined as in (1.3). If  $X = 0$ , then  $\tilde{J}_{\omega_0}$  and  $\tilde{I}_{\omega_0}$  are just the original Aubin–Yau function  $I_{\omega_0}$  and  $J_{\omega_0}$ .

**Proposition 2.1** ([5]).  *$I_{\omega_0}$ ,  $J_{\omega_0}$ ,  $\tilde{I}_{\omega_0}$  and  $\tilde{J}_{\omega_0}$  are positive on  $\mathcal{H}_X(M, \omega_0)$ . There exist positive constants  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$ , where  $C_1$  and  $C_2$  is less than 1 such that for any  $\varphi \in \mathcal{H}_X(M, \omega_0)$ , such that*

$$(2.1) \quad 0 \leq C_3 I_{\omega_0}(\varphi) \leq C_1 \tilde{I}_{\omega_0}(\varphi) \leq \tilde{I}_{\omega_0}(\varphi) - \tilde{J}_{\omega_0}(\varphi) \leq C_2 \tilde{I}_{\omega_0}(\varphi) \leq C_4 I_{\omega_0}(\varphi).$$

Assume  $\omega_\phi$  be another Kähler form in  $[\omega_0]$ , we have

$$(2.2) \quad |I_{\omega_\phi}(\varphi - \phi) - I_{\omega_0}(\varphi)| \leq (n+1) \text{OSC}(\phi),$$

for all  $\varphi \in \mathcal{H}_X(M, \omega_0)$ .

We define twisted Mabuchi  $K$ -energy  $\tilde{\mu}_{\omega_0, \eta}$  and Ding–functional  $\tilde{F}_{\omega_0, \eta}$  on the function space  $\mathcal{H}_X(M, \omega_0)$  as follow:

$$\tilde{\mu}_{\omega_0, \eta}(\varphi) = \frac{\sqrt{-1}n}{2\pi V} \int_0^1 \int_M e^{\theta_X(\omega_{\varphi_t})} \partial(h_{\omega_{\varphi_t}} - \theta_X(\omega_{\varphi_t})) \wedge \bar{\partial} \dot{\varphi}_t \omega_{\varphi_t}^{n-1} \wedge dt,$$

and

$$\tilde{F}_{\omega_0, \eta}(\varphi) = \tilde{J}_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega_0^n - \frac{1}{\beta} \log\left(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta\varphi} \omega_0^n\right).$$

For convenience, we define the functional

$$\hat{F}_{\omega_0} = \tilde{J}_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega_0^n.$$

**Proposition 2.2.** *The functional  $\tilde{F}_{\omega_0, \eta}$ ,  $\tilde{\mu}_{\omega_0, \eta}$  and  $\hat{F}_{\omega_0}$  are well-defined, i.e. independent of the choice of the path  $\{\varphi_t\}$ . Furthermore, all of them satisfy the cocycle property.*

**Lemma 2.3.** *For any  $\varphi \in \mathcal{H}_X(M, \omega_0)$ ,*

$$(2.3) \quad \begin{aligned} \tilde{\mu}_{\omega_0, \eta}(\varphi) = & \beta \tilde{F}_{\omega_0, \eta}(\varphi) + \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n \\ & - \frac{1}{V} \int_M (h_{\omega_\varphi} - \theta_X - X(\varphi)) e^{\theta_X + X(\varphi)} \omega_\varphi^n. \end{aligned}$$

Furthermore, we have

$$(2.4) \quad \tilde{\mu}_{\omega_0, \eta}(\varphi) \geq \beta \tilde{F}_{\omega_0, \eta}(\varphi) + \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n.$$

**Lemma 2.4.** *Assume that  $\eta_1 - \eta_2 = \sqrt{-1}\partial\bar{\partial}f$ , for some smooth function  $f$ , then*

$$|\tilde{\mu}_{\omega_0, \eta_1}(\varphi) - \tilde{\mu}_{\omega_0, \eta_2}(\varphi)| \leq \text{OSC}(f).$$

### 3. EXISTENCE AND UNIQUENESS RESULT FOR THE TWISTED KÄHLER-RICCI SOLITON

As in Kähler-Einstein and Kähler-Ricci soliton cases, finding twisted Kähler-Ricci soliton can be reduced to solving the complex Monge-Ampère equation (1.5). To solve (1.5), we use the continuity method. We consider a family of complex Monge-Ampère equations,

$$(3.1) \quad \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n} = e^{h_{\omega_0} - \beta t\varphi - \theta_X - X(\varphi)}$$

i.e.

$$(3.2) \quad \text{Ric}\omega_\varphi = \beta t\omega_\varphi + (\beta - \beta t)\omega_0 + \eta + L_X\omega_\varphi$$

and set

$$S = \{t \in [0, 1] \mid (3.1) \text{ is solvable for } t\}.$$

For our convenience, we assume  $\eta = (1 - \beta)(\omega_0 + \sqrt{-1}\partial\bar{\partial}f_\eta)$ , where  $f_\eta$  is a smooth function such that  $\sup_M f_\eta = 0$ . By [33] and the property of  $\eta$ , we know that (3.1) is solvable for  $t = 0$ , thus  $S$  is not empty. If we can prove that  $S$  is both open and close, then we must have  $S = [0, 1]$  and hence the complex Monge-Ampère equation (1.5) is solvable. In the proof of the openness and closeness of  $S$ , we need the assumption that  $\eta$  is semipositive. The key point is that the semipositivity of  $\eta$  will lead to a lower bound of the Ricci curvature by a positive constant. Then we can apply the implicit function theorem to prove the openness and obtain a lower bound of the Green's function for the weighted Laplace, which is crucial to get  $C^0$

estimate. We will follow methods of [5], [22] and [32] to obtain the openness and closeness. First, we present the following proposition for further discussion.

**Proposition 3.1.** *Let  $0 < \tau \leq 1$ , and suppose that (3.1) is solvable at  $t = \tau$ . We have the following,*

- (1). *If  $0 < \tau < 1$ , there exists some  $\varepsilon > 0$  such that (3.1) can be solvable uniquely for  $t \in (\tau - \varepsilon, \tau + \varepsilon) \cap (0, 1)$ .*
- (2).  *$S$  is also open near  $t = 0$ , i.e.  $\exists$  a small positive number  $\varepsilon$  such that there is a smooth family of solutions of (3.1) for  $t \in (0, \varepsilon)$ .*
- (3). *If  $\eta$  is strictly positive at a point,  $S$  is open near 1, i.e. (3.1) is solvable for  $t \in (1 - \varepsilon, 1]$  for some small positive  $\varepsilon$ .*

*Proof.* For  $2 \leq \gamma \in \mathbb{Z}^+$  and  $0 < \alpha < 1$ , we define

$$\mathcal{H}_X^{\gamma, \alpha}(\omega_0) = \{\phi \in C^{\gamma, \alpha}(M) \mid \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0, \text{ and } \text{Im } X(\phi) = 0\},$$

and

$$\mathcal{W}_X^{\gamma, \alpha} = \{\phi \in C^{\gamma, \alpha}(M) \mid \text{Im } X(\phi) = 0\}.$$

It is easy to see that the tangent space of  $\mathcal{H}_X^{\gamma, \alpha}(\omega_0)$  is  $\mathcal{W}_X^{\gamma, \alpha}$ . Consider the operator  $\Psi : \mathcal{H}_X^{\gamma, \alpha}(\omega_0) \times [0, 1] \rightarrow C^{\gamma-2, \alpha}(M)$  defined by

$$\Psi(\varphi, t) := \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n} - h_{\omega_0} + t\beta\varphi + X(\varphi).$$

The linearized operator of  $\Psi$  at  $(t, \varphi)$  is given by

$$L_{t, \varphi}(\phi) = \Delta_{\omega_\varphi} \phi + t\beta\phi + X(\phi),$$

for  $\phi \in \mathcal{W}_X^{\gamma, \alpha}$ . Now we prove that  $L_{t, \varphi}$  is invertible. Assume  $\lambda_{1, t}$  is the first eigenvalue of  $L_{t, \varphi}$ , and  $\phi$  is an eigenfunction of  $L_{t, \varphi}$  with respect to  $\lambda_{1, t}$ , i.e.  $L_{t, \varphi}(\phi) = -\lambda_{1, t}\phi$ . Applying the Bochner formula and equation (3.1), we get that,

$$\begin{aligned} & \lambda_{1, t} \int_M |\nabla_{\omega_\varphi} \phi|_{\omega_\varphi}^2 e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ &= - \int_M \langle \nabla_{\omega_\varphi} (\Delta_{\omega_\varphi} \phi + t\beta\phi + X(\phi)), \nabla_{\omega_\varphi} \phi \rangle_{\omega_\varphi} e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ (3.3) \quad &= \frac{1}{2} \int_M (\beta(1-t)\omega_0 + \eta)(\nabla_{\omega_\varphi} \phi, J(\nabla_{\omega_\varphi} \phi)) e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ & \quad + \int_M \left| \nabla_{\omega_\varphi}^{(1,0)} \nabla_{\omega_\varphi}^{(1,0)} \phi \right|_{\omega_\varphi}^2 e^{\theta_X + X(\varphi)} \omega_\varphi^n \end{aligned}$$

In the case  $0 \leq \tau < 1$ , we have

$$\begin{aligned} & \lambda_{1, \tau} \int_M |\nabla_{\omega_\varphi} \phi|_{\omega_\varphi}^2 e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ & \geq \frac{(1-\tau)\beta}{2} \int_M \omega_0(\nabla_{\omega_\varphi} \phi, J(\nabla_{\omega_\varphi} \phi)) e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ & > 0, \end{aligned}$$

which implies that  $\lambda_{1, \tau} > 0$ , i.e.  $L_{\tau, \varphi}$  is invertible. And consequently, the first and second statements of the proposition hold.

When  $\tau = 1$ , if  $\lambda_{1, \tau} = 0$ , then

$$\frac{1}{2} \int_M \eta(\nabla_{\omega_\varphi} \phi, J(\nabla_{\omega_\varphi} \phi)) e^{\theta_X + X(\varphi)} \omega_\varphi^n + \int_M \left| \nabla_{\omega_\varphi}^{(1,0)} \nabla_{\omega_\varphi}^{(1,0)} \phi \right|_{\omega_\varphi}^2 e^{\theta_X + X(\varphi)} \omega_\varphi^n = 0.$$

Since  $\eta$  is smooth and strictly positive at some point  $p$ , we get that  $\nabla_{\omega_\varphi}^{(1,0)}\phi = 0$  in a neighborhood of  $p$ , which implies that  $\nabla_{\omega_\varphi}^{(1,0)}\phi = 0$ . So  $L_{1,\varphi}$  is invertible. And the last statement holds.  $\square$

Next, we prove the closeness of  $S$ . Let  $\{\varphi_t\}$  be a smooth family of solution of (3.1) for  $t \in (0, 1]$ . Along the path  $\{\varphi_t\}$ , we have

$$(3.4) \quad \int_M e^{h_{\omega_0} - \beta t \varphi_t} \omega_0^n = \int_M e^{\theta_X + X(\varphi_t)} \omega_{\varphi_t}^n = V.$$

Differentiating (3.4) with respect to  $t$ , we have

$$(3.5) \quad \int_M \varphi_t e^{h_{\omega_0} - \beta t \varphi_t} \omega_0^n = - \int_M t \dot{\varphi}_t e^{h_{\omega_0} - \beta t \varphi_t} \omega_0^n.$$

**Proposition 3.2.** *Let  $\{\varphi_t\}$  be a smooth family of solution of (3.1) for  $t \in (0, 1]$ . Then*

$$\hat{F}_{\omega_0}(\varphi_t) = -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_0}(\varphi_s) - \tilde{J}_{\omega_0}(\varphi_s)) dt.$$

Furthermore, we have:

**Lemma 3.3.** *Let  $\{\varphi_t\}$  be a smooth family of solution of (3.1) for  $t \in (0, 1]$ . We have*

$$\frac{d}{dt}(\tilde{I}_{\omega_0}(\varphi_t) - \tilde{J}_{\omega_0}(\varphi_t)) \geq 0,$$

i.e.  $\tilde{I}_{\omega_0}(\varphi_t) - \tilde{J}_{\omega_0}(\varphi_t)$  is nondecreasing with respect to  $t$  along  $\{\varphi_t\}$ .

Before proving the closeness of  $S$ , we recall two useful estimates which have been applied in [5] and [33].

**Lemma 3.4** ([5]). *Let  $\varphi \in \mathcal{H}_X(M, \omega_0)$ . Suppose that*

$$\text{Ric}(\omega_\varphi) - L_X \omega_\varphi \geq \lambda \omega_\varphi,$$

and

$$\Delta_{\omega_\varphi} \theta_X(\omega_\varphi) \leq k,$$

for some positive number  $\lambda$  and  $k$ . Then, there are uniform constants  $C_1, C_2$  depending only on  $\lambda$  and  $k$  such that the Green function  $G$  with respect to the operator  $\Delta_{\omega_\varphi} + X$  is bounded from below by  $C_1$  and the following estimate for  $\sup_M(-\varphi)$  holds,

$$\sup_M(-\varphi) \leq \frac{1}{V} \int_M (-\varphi) e^{\theta_X(\omega_\varphi)} \omega_\varphi^n + C_2.$$

**Lemma 3.5** ([33]). *For any  $\varphi \in \mathcal{H}_X(M, \omega_0)$ , there exists a uniform constant  $C$  independent of  $\varphi$ , such that  $|X(\varphi)| \leq C$ .*

Next, we consider the closeness of  $S$ .

**Proposition 3.6.** *Let  $\varphi = \varphi_t (t \geq t_0 > 0)$  be a solution of (3.1) at  $t$ . Suppose that  $\tilde{\mu}_{\omega_0, \eta}$  (or  $\tilde{F}_{\omega_0, \eta}$ ) is proper, then, the  $C^0$ -norm of  $\varphi$  is bounded depending only on  $X, t_0$ , the properness and the geometry of  $(M, \omega_0)$ .*



*Proof.* Following from Lemma 2.3, we just prove the case when  $\tilde{\mu}_{\omega_0, \eta}$  is proper. To do this we first give an estimate of the oscillation of  $\varphi$ .

Let  $\theta'_X = \theta_X(\omega_\varphi) = \theta_X + X(\varphi)$ . By the equation

$$L_X Ric(\omega) = -\sqrt{-1}\partial\bar{\partial} \Delta_\omega \theta_X(\omega),$$

and the maximal principle, we have

$$\Delta_{\omega_\varphi} \theta'_X = -\beta \theta'_X - (1-\beta)\theta_X - (1-\beta)X(f_\eta) - X(h_{\omega_{\varphi_t}}) + c_t$$

for some constant  $c_t$ . By using the maximal principle to (3.2), we have

$$h_{\omega_\varphi} = \theta'_X - (1-t)\beta\varphi + c'_t,$$

where  $c'_t$  is some constant, which implies that

$$(3.6) \quad \Delta_{\omega_\varphi} \theta'_X = -\beta \theta'_X - (1-\beta)\theta_X - (1-\beta)X(f_\eta) - X(\theta'_X) + (1-t)\beta X(\varphi) + c_t.$$

Applying the maximal principle and (3.6), we get that

$$(3.7) \quad c_t \leq \beta \|\theta'_X\|_{C^0} + (1-\beta)\|\theta_X\|_{C^0} + (1-\beta)\|X(f_\eta)\|_{C^0} + \beta(1-t)\|X(\varphi)\|_{C^0}.$$

So we have

$$\begin{aligned} \Delta_{\omega_\varphi} \theta'_X &\leq -\|X\|_{\omega_\varphi}^2 + (2 + \beta(1-t))\|\theta_X\|_{C^0} + (2\beta + 2\beta(1-t))\|X(\varphi)\|_{C^0} \\ &\quad + 2(1-\beta)\|X(f_\eta)\|_{C^0} \\ &\leq k \end{aligned}$$

for some uniform constant  $k$  by (3.6), (3.7) and Lemma 3.3.

Furthermore, we have that

$$Ric(\omega_\varphi) - L_X \omega_\varphi = \beta t \omega_\varphi + \beta(1-t)\omega_0 + \eta \geq \beta t_0 \omega_\varphi.$$

By Lemma 3.4 with  $\lambda = \beta t_0$ , we get that

$$(3.8) \quad \sup_M (-\varphi) \leq \frac{1}{V} \int_M (-\varphi) e^{\theta'_X} \omega_\varphi^n + C'_1,$$

for some uniform constant  $C'_1$  depending only on  $X$ ,  $t_0$ .

On the other hand, by using the Green formula for the Laplace with respect to  $\tilde{\omega}$  satisfying that

$$\tilde{\omega}^n = e^{\theta_X} \omega_0^n,$$

we get that

$$(3.9) \quad \sup_M \varphi \leq \frac{1}{V} \int_M \varphi e^{\theta_X} \omega_0^n + C'_2$$

for some uniform constant  $C'_2$  (cf. Lemma 5.3 in [25]). Hence, combining (3.8) and (3.9), we have

$$\text{OSC}_M \varphi \leq \frac{1}{V} \int_M \varphi (e^{\theta_X} \omega_0^n - e^{\theta'_X} \omega_\varphi^n) + C'_3,$$

for some uniform constant  $C'_3$ .

Next we show that

$$\tilde{I}_{\omega_0}(\varphi) = \frac{1}{V} \int_M \varphi (e^{\theta_X} \omega_0^n - e^{\theta'_X} \omega_\varphi^n)$$

is uniformly bounded from above under the condition that  $\tilde{\mu}_{\omega_0, \eta}$  is proper.

By the equation (3.1) and the definition of  $h_\omega$ , we get that

$$\begin{aligned}
 h_{\omega_\varphi} &= h_{\omega_0} - \log \frac{\omega_\varphi^n}{\omega_0^n} - \beta\varphi - \log\left(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta\varphi} \omega_0^n\right) \\
 (3.10) \quad &= h_{\omega_0} - (h_{\omega_0} - \beta t\varphi - \theta'_X) - \beta\varphi - \log\left(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta\varphi} \omega_0^n\right) \\
 &= \theta'_X - \beta(1-t)\varphi - \log\left(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta\varphi} \omega_0^n\right).
 \end{aligned}$$

Proposition 2.1, Proposition 3.2 and equation (3.10) imply that

$$\begin{aligned}
 \tilde{\mu}_{\omega_0, \eta}(\varphi) &= \beta \tilde{F}_{\omega_0, \eta}(\varphi) + \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n - \frac{1}{V} \int_M (h_{\omega_\varphi} - \theta'_X) e^{\theta'_X} \omega_\varphi^n \\
 &= \beta \hat{F}_{\omega_0}(\varphi) + \frac{\beta(1-t)}{V} \int_M \varphi e^{\theta'_X} \omega_\varphi^n + \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n \\
 &= \beta(1-t)(\tilde{J}_{\omega_0}(\varphi) - \tilde{I}_{\omega_0}(\varphi)) + \beta t \hat{F}_{\omega_0}(\varphi) + \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n \\
 &\leq \frac{1}{V} \int_M (h_{\omega_0} - \theta_X) e^{\theta_X} \omega_0^n \\
 &\leq C'_4,
 \end{aligned}$$

$\tilde{I}_{\omega_0}(\varphi)$  is uniform bounded from above, since  $\tilde{\mu}_{\omega_0, \eta}$  is proper, and Proposition 2.1. So we get the  $C^0$ -estimate for the twisted Kähler-Ricci soliton.  $\square$

By Yau's estimate [31] or Zhu's estimate [33] for complex Monge-Ampère equations, the  $C^0$ -estimate implies the  $C^{2, \alpha}$ -estimate and the elliptic Schauder estimates give the higher order estimates. We get the closeness of  $S$ , and the existence of twisted Kähler-Ricci soliton with respect to  $\eta$ .

*Remark 3.1.* As a consequence of the argument above and the argument of Bando-Mabuchi, we can get easily that if the modified K-energy  $\tilde{\mu}_{\omega, X}$  in [5] is bounded from below, then  $R(X) = 1$ .

Following the idea of [3], we consider the uniqueness of the twisted Kähler-Ricci soliton with respect to a smooth semipositive  $(1, 1)$ -form  $\eta$  which is strictly positive at one point.

**Theorem 3.7.** *If  $\eta$  is a smooth semipositive  $(1, 1)$ -form cohomology to  $(1 - \beta)\omega_0$  and  $\eta$  is strictly positive at one point, the solution to (1.5) is unique.*

*Proof.* We assume that  $\omega_1 = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_1$  and  $\omega_2 = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_2$  are two different solutions to (1.5). It is easy to see that  $\varphi_1$  and  $\varphi_2$  are stationary points of  $\tilde{F}_{\omega_0, \eta}$ . According to page 32 of [3], we assume that  $\{\phi_t | 0 \leq t \leq 1\}$  is a curve of  $C^1$ -geodesics connecting  $\varphi_1$  and  $\varphi_2$ , and  $\text{Im } X(\phi_t) = 0$ .

Direct computation shows that  $\hat{F}_{\omega_0}(\phi_t)$  is linear with respect to  $t$ . Theorem 1.1 of [3] shows that

$$-\frac{1}{\beta} \log\left(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta\phi_t} \omega_0^n\right)$$

is convex with respect to  $t$ . So is  $\tilde{F}_{\omega_0, \eta}(\phi_t)$ . Since  $\varphi_1$  and  $\varphi_2$  are stationary points of  $\tilde{F}_{\omega_0, \eta}$ , we get that  $\tilde{F}_{\omega_0, \eta}(\phi_t)$  is actually linear with respect to  $t$ . So

$-\frac{1}{\beta} \log(\frac{1}{V} \int_M e^{h_{\omega_0} - \beta \phi_t} \omega_0^n)$  is linear in  $t$ . Theorem 6.2 of [3] implies that there exists a holomorphic vector field  $V_t$  on  $M$  with flow  $F_t$  such that

$$(3.11) \quad F_t^*(\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_0 = \omega_1$$

$$(3.12) \quad i_{V_t} \eta = 0.$$

However, the semi-positiveness and strict positiveness of  $\eta$  at some point implies that  $V_t = 0$ . So  $F_t$  is an identity. The equation (3.11) shows that  $\omega_1 = \omega_2$ , which is a contradiction.  $\square$

#### 4. A MOSER-TRUDINGER TYPE INEQUALITY

Let  $\omega_0 \in 2\pi c_1(M)$  be a Kähler form on  $M$ , satisfying that

$$\begin{cases} Ric(\omega_0) - L_X \omega_0 \geq (\beta - \varepsilon) \omega_0 + \eta \\ |X(h_{\omega_0} - \theta_X)| \leq \varepsilon c_1 \end{cases}$$

for some positive number  $\varepsilon$  and  $c_1$ , where  $\theta_X = \theta_X(\omega_0)$ . Similar to [5], we define  $a(\omega)$  to be the maximal constant such that,

$$\forall 0 < r < 1, \text{ and } x \in M, \int_{B_r(x)} e^{\theta_X(\omega)} \omega^n \geq a(\omega) r^{2n},$$

where  $B_r(x)$  is the geodesic ball in  $M$  of radio  $r$  centering  $x$  with respect to  $\omega$ . And

$$\lambda_1(\omega) = \inf_{\substack{v \in C^\infty(M) \\ \nabla_\omega v \neq 0}} \frac{\int_M |\nabla_\omega v|_\omega^2 e^{\theta_X(\omega)} \omega^n}{\int_M v^2 e^{\theta_X(\omega)} \omega^n - (\int_M v e^{\theta_X(\omega)} \omega^n)^2},$$

is the first eigenvalue of the operator  $\Delta_\omega + X$ . For two Kähler forms  $\omega_1, \omega_2 \in \mathcal{K}_X(\omega_0)$  satisfying

$$\frac{1}{2} \omega_2 \leq \omega_1 \leq 2\omega_2,$$

we have the following estimates

$$(4.1) \quad \lambda_1(\omega_1) \geq 2^{-2n-1} e^{-2C(\omega_0, X)} \lambda_1(\omega_2),$$

$$(4.2) \quad a(\omega_1) \geq 2^{-2n} e^{-C(\omega_0, X)} a(\omega_2),$$

where  $C(\omega_0, X)$  is the constant appeared in Lemma 3.5. Similar to [5], [22] and [32], we have the smoothing lemma as follow:

**Lemma 4.1** (Smoothing lemma).  $\omega_1$  is the solution of

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \eta + \beta \omega_t + L_X \omega_t$$

at  $t = 1$  with initial data  $\omega_0$ . And  $\omega_t = \omega_0 + \sqrt{-1} \partial \bar{\partial} u_t$ ,  $u_0 = 0$ . The following inequalities hold:

1.  $\|u_1\|_{C^0(M)} \leq \frac{\varepsilon^\beta}{\beta} \|h_{\omega_0} - \theta_X\|_{C^0(M)},$
2.  $\|h_{\omega_1} - \theta_X(\omega_1)\|_{C^{0, \frac{1}{2}}(\omega_1)} \leq 4C(n, c_1, a(\omega_1), \lambda_1(\omega_1))(1 + \|h_{\omega_0} - \theta_X\|_{C^0(M)}) \varepsilon^{\frac{1}{4(n+1)}},$

where  $C(n, c_1, a(\omega_1), \lambda_1(\omega_1)) = e^\beta (1 + \sqrt{\frac{2V(c_1+n)}{a(\omega_1)\lambda_1(\omega_1)}}).$

Using the smoothing lemma and discussion similar to [5], [19] and [32], we can establish a Moser–Trudinger type inequality for the twisted Ding–functional  $\tilde{F}_{\omega_{TKS}, \eta}$  or  $K$ –energy  $\tilde{\mu}_{\omega_{TKS}, \eta}$ . Specially, the Moser–Trudinger inequality here is linear form. And for the readers’ convenience, we give the details of the proof:

**Theorem 4.2.** *Let  $(M, \omega_0)$  be a Kähler manifold and  $L_{(\text{Im } X)}\omega_0 = 0$  where  $X$  is a holomorphic vector field on  $M$ . Assuming that  $\eta \in (1-\beta)\mathcal{K}_X^0(\omega_0)$  is strictly positive at one point. If there exists a twisted Kähler–Ricci soliton metric  $\omega_{TKS} \in \mathcal{K}_X(\omega_0)$ , then there exist uniform positive constants  $C_1, C_2$  depending only on  $\eta, \beta$  and the geometry of  $(M, \omega_{TKS})$  such that*

$$\tilde{F}_{\omega_{TKS}, \eta}(\phi) \geq C_1 J_{\omega_{TKS}}(\phi) - C_2$$

for all  $\phi \in \mathcal{H}_X(M, \omega_{TKS})$ .

*Proof.* For any  $\phi \in \mathcal{H}_X(M, \omega_{TKS})$ , we denote  $\omega_g = \omega_{TKS} + \sqrt{-1}\partial\bar{\partial}\phi$ . We consider the following Monge–Ampère equations with parameter  $t \in [0, 1]$  with the notions  $\theta_X = \theta_X(\omega_g)$ :

$$(4.3) \quad \begin{cases} \frac{(\omega_g + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega_g^n} = h_{\omega_g} - \theta_X - X(\phi) - \beta t\phi \\ \omega_g + \sqrt{-1}\partial\bar{\partial}\phi > 0. \end{cases}$$

Clearly,  $-\phi$  is a solution of (4.3) at  $t = 1$  modulo a constant. Let

$$T = \{t \in [0, 1] \mid (4.3) \text{ is solvable at } t\}.$$

By the assumption that  $\eta$  is strictly positive at one point and the fact that  $\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)$  is nondecreasing in  $t$ , we get that  $(0, 1] \subset T$ , i.e. for any  $t \in (0, 1]$ , there is a solution  $\phi_t$  for (4.3) at  $t$ . We assume that  $\phi_t$  is the solution of (3.2) at  $t$ , with  $\phi_1 = -\phi + C$  and denote that  $\omega_t = \omega_g + \sqrt{-1}\partial\bar{\partial}\phi_t$ . Consequently, the  $C^3$ –norm of  $\phi_t$  for  $t \geq \frac{1}{2}$  is uniformly bounded by  $\phi, \omega_{TKS}, n, \eta$ , and  $X$ .

Since  $\omega_t$  is a solution to

$$\text{Ric}(\omega_t) = \beta t\omega_t + (\beta - \beta t)\omega_g + \eta + L_X\omega_t,$$

the maximal principle implies that

$$h_{\omega_t} - \theta_X(\omega_t) = -\beta(1-t)\phi_t + c'_t,$$

where  $c'_t$  is determined by

$$\int_M e^{c'_t - \beta(1-t)\phi_t + \theta_X(\omega_t)} \omega_t^n = \int_M e^{h_{\omega_t}} \omega_t^n = \int_M \omega_t^n = \int_M e^{\theta_X(\omega_t)} \omega_t^n.$$

In particular, we get that  $c'_t - \beta(1-t)\phi_t$  changes sign, which implies that  $|c'_t| \leq \beta(1-t)\|\phi_t\|_{C^0(M)}$ , and consequently,

$$\|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0(M)} \leq 2\beta(1-t)\|\phi_t\|_{C^0(M)}.$$

On the other hand, the Kähler metric  $\omega_t$  satisfies that

$$\begin{cases} \text{Ric}\omega_t - L_X\omega_t = \beta t\omega_t + \eta + \beta(1-t)\omega_g \geq \beta t\omega_t + \eta \\ |X(h_{\omega_t} - \theta_X(\omega_t))| = \beta(1-t)|X(\phi_t)| \leq 2\beta(1-t)C(\omega_{TKS}, X). \end{cases}$$

With  $\varepsilon = \beta - \beta t$ ,  $c_1 = 2C(\omega_{TKS}, X)$ , we know that  $\omega_t$  satisfies that

$$(4.4) \quad \begin{cases} \text{Ric}(\omega_t) - L_X\omega_t \geq (\beta - \varepsilon)\omega_t + \eta \\ |X(h_{\omega_t} - \theta_X(\omega_t))| \leq \varepsilon c_1. \end{cases}$$

Let  $\omega_t$  be the initial metric in the flow we have considered in last section. Applying the smoothing lemma, we obtain a Kähler form  $\omega'_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u_t$  satisfying that

$$\begin{aligned} \|u_t\|_{C^0(M)} &\leq \frac{e^\beta}{\beta} \|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0(M)} \leq 2e^\beta(1-t) \|\varphi_t\|_{C^0(M)} \\ \|h_{\omega'_t} - \theta_X(\omega'_t)\|_{C^{0,\frac{1}{2}}(\omega'_t)} &\leq 4C(n, c_1, a(\omega'_t), \lambda_1(\omega'_t))(1 + \|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0(M)})(\beta - \beta t)^{\frac{1}{4n+4}} \\ &\leq 8C(n, c_1, a(\omega'_t), \lambda_1(\omega'_t))(1 + (1-t)\|\varphi_t\|_{C^0(M)})(\beta - \beta t)^{\frac{1}{4n+4}}. \end{aligned}$$

As before, there exists  $\varphi'_t$  such that  $\omega_{TKS} = \omega'_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$  and

$$-\log \frac{\omega_{TKS}^n}{\omega_t'^n} = -h_{\omega'_t} + \beta\varphi'_t + \theta_X(\omega'_t) + X(\varphi'_t).$$

It follows from the maximal principle that

$$(4.5) \quad \varphi_t = \varphi_1 - \varphi'_t + u_t + \mu_t,$$

where  $\mu_t$  is a constant. The normalization  $\int_M e^{h_{\omega'_t}} \omega_t'^n = \int_M \omega_t'^n$  implies that

$$|\mu_t| \leq 2(1 + e^\beta)(1-t) \|\varphi_t\|_{C^0(M)}.$$

Next, we give an estimate of  $\|\varphi'_t\|_{C^0(M)}$  while  $t$  is sufficiently closed to 1. We denote the operator

$$\Xi : \mathcal{H}_X^{2,\frac{1}{2}}(M, \omega_{TKS}) \times [0, 1] \rightarrow C^{0,\frac{1}{2}}(M)$$

by

$$\Xi(\varphi', t) = \log \frac{(\omega_{TKS} - \sqrt{-1}\partial\bar{\partial}\varphi')^n}{\omega_{TKS}^n} + h_{\omega'_t} - \beta\varphi' - \theta_X(\omega'_t) - X(\varphi').$$

Obviously,  $\Xi(0, 1) = 0$ , and the linearization operator of  $\Xi$  at  $\varphi' = 0$  is

$$-\Delta_{\omega_{TKS}} - X - \beta,$$

which is invertible under the assumption that  $\eta$  is strictly positive at one point. So  $\Xi$  is invertible for  $t$  sufficiently closed to 1, i.e.  $\exists \sigma > 0$  if

$$(4.6) \quad \|h_{\omega'_t} - \theta_X(\omega'_t)\|_{C^{0,\frac{1}{2}}(\omega_{TKS})} \leq \sigma$$

then,

$$(4.7) \quad \exists! \varphi'_t, \Xi\varphi'_t = 0, \text{ and } \|\varphi'_t\|_{C^{2,\frac{1}{2}}(\omega_{TKS})} \leq C_0\sigma,$$

where  $C_0 = C(n, c_1, 2^{-2n}e^{-C(X, \omega_{TKS})}a(\omega_{TKS}), 2^{-2n-1}e^{-2C(X, \omega_{TKS})}\lambda_1(\omega_{TKS})) + 1$  with notations that  $c_1$  is the constant in (4.4) and  $C(X, \omega_{TKS})$  in Lemma 3.4. And we choose  $\sigma$  small such that  $C_0\sigma < \frac{1}{4}$  and  $\frac{1}{\beta}(\frac{\sigma}{32C_0})^{4n+4} \leq \frac{1}{12e^\beta}$ .

For convenience, we do some computation to  $\tilde{F}_{\omega_{TKS}, \eta}(\phi)$ :

$$\begin{aligned} \tilde{F}_{\omega_{TKS}, \eta}(\phi) &= -\tilde{F}_{\omega_g, \eta}(\varphi_1) = -\hat{F}_{\omega_g}(\varphi_1) \\ &= \int_0^1 (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) dt \\ &\geq c(1-t)\tilde{J}_{\omega_g}(\varphi_t). \end{aligned}$$

$$\begin{aligned}
\tilde{J}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_1) &= \int_M \varphi_t e^{\theta x} \omega_g^n - \int_M \varphi_1 e^{\theta x} \omega_g^n + \hat{F}_{\omega_g}(\varphi_t) - \hat{F}_{\omega_g}(\varphi_1) \\
&= \int_M (\varphi_t - \varphi_1) e^{\theta x} \omega_g^n + \hat{F}_{\omega_{TKS}}(\varphi_t - \varphi_1) \\
&= \int_M (\varphi_t - \varphi_1) (e^{\theta x} \omega_g^n - e^{\theta x(\omega_{\varphi_1})} \omega_{\varphi_1}^n) + \tilde{J}_{\omega_{TKS}}(\varphi_t - \varphi_1) \\
&\geq \int_M (\varphi_t - \varphi_1) (e^{\theta x} \omega_g^n - e^{\theta x(\omega_{\varphi_1})} \omega_{\varphi_1}^n) \\
&\geq -\text{OSC}_M(\varphi_t - \varphi_1).
\end{aligned}$$

Hence,  $\tilde{J}_{\omega_g}(\varphi_t) \geq cI_{\omega_g}(\varphi_1) - \text{OSC}_M(\varphi_t - \varphi_1)$ . Consequently,

$$\begin{aligned}
(4.8) \quad \tilde{F}_{\omega_{TKS}, \eta}(\phi) &\geq c(1-t)I_{\omega_g}(\varphi_1) - C(1-t)\text{OSC}_M(\varphi_t - \varphi_1) \\
&= c(1-t)I_{\omega_{TKS}}(\phi) - C(1-t)\text{OSC}_M(\varphi_t - \varphi_1).
\end{aligned}$$

To prove the inequality required, we will discuss case by case.

**Case1:** Existing  $t_0$ , such that  $((1-t_0)\beta)^{\frac{1}{4(n+1)}} \leq \frac{\sigma}{32C_0}$ , and

$$\forall t \in (t_0, 1], (1-t)\|\varphi_t\|_{C^0(M)} ((1-t)\beta)^{\frac{1}{4(n+1)}} < \frac{\sigma}{32C_0},$$

$$(1-t_0)\|\varphi_{t_0}\|_{C^0(M)} ((1-t_0)\beta)^{\frac{1}{4(n+1)}} = \frac{\sigma}{32C_0}.$$

Then we claim that  $\|\varphi'_t\|_{C^{2, \frac{1}{2}}}(\omega_{TKS}) < \frac{1}{4}$  for  $t \in (t_0, 1]$ . If not, existing  $t_1 \in (t_0, 1]$  such that  $\|\varphi'_{t_1}\|_{C^{2, \frac{1}{2}}}(\omega_{TKS}) = \frac{1}{4}$ . So we have  $\frac{1}{2}\omega_{TKS} \leq \omega'_{t_1} \leq 2\omega_{TKS}$ . Associated with the inequality above, we get that:

$$\left\| h_{\omega'_{t_1}} - \theta_X(\omega'_{t_1}) \right\|_{C^{0, \frac{1}{2}}(\omega_{TKS})} < \sigma$$

But (4.6) implies that  $\|\varphi'_{t_1}\|_{C^{2, \frac{1}{2}}(\omega_{TKS})} \leq C_0\sigma < \frac{1}{4}$ , which is a contradiction. The equation (4.5) implies that for all  $t \in [t_0, 1]$

$$\begin{aligned}
(4.9) \quad \|\varphi_t - \varphi_1\|_{C^0(M)} &\leq \frac{1}{4} + \|u_t\|_{C^0(M)} + |\mu_t| \\
&\leq \frac{1}{4} + 6e^\beta(1-t)\|\varphi_t\|_{C^0(M)},
\end{aligned}$$

Since  $1-t \leq 1-t_0 \leq \frac{1}{12e^\beta}$ , then we have that

$$\frac{1}{2}\|\varphi_t\|_{C^0(M)} - \frac{1}{4} \leq \|\varphi_1\|_{C^0(M)} \leq 2\|\varphi_t\|_{C^0(M)} + \frac{1}{4}.$$

Further more, associated with (4.8) we can get that

$$\begin{aligned}
\tilde{F}_{\omega_{TKS}, \eta}(\phi) &\geq c(1-t)I_{\omega_{TKS}}(\phi) - Ce^\beta(24\|\varphi_1\|_{C^0(M)} - 6)(1-t)^2 - \frac{C}{2}(1-t) \\
&\geq c(1-t)I_{\omega_{TKS}}(\phi) - 24Ce^\beta(1-t)^2\text{OSC}_M\phi - C(1-t).
\end{aligned}$$

In case  $\text{OSC}_M\phi \leq \hat{C}(1+I_{\omega_{TKS}}(\phi))$ , we get  $\tilde{F}_{\omega_{TKS}, \eta}(\phi) \geq c(1-t)I_{\omega_{TKS}}(\phi) - C$  for some uniform constant  $c$  and  $C$ . Now choosing  $t = t_0$ , by the definition of  $t_0$ , we get that

$$(1-t_0)^{\frac{4n+5}{4n+4}}I_{\omega_{TKS}}(\phi) \geq \frac{\beta^{4n+4}\sigma}{32C_0} - \left(\frac{1}{2} + 2\hat{C}\right)\left(\frac{\beta^{4n+4}\sigma}{32C_0}\right)^{\frac{4n+5}{4n+4}} > C(\sigma) > 0,$$

therefore,

$$\tilde{F}_{\omega_{TKS},\eta}(\phi) \geq CI_{\omega_{TKS}}^{\frac{1}{4n+5}}(\phi) - C'.$$

By the  $C^0$ -estimate in last section, we know that there exists a uniform constant  $\hat{C}$  depending only on  $X$ ,  $t_0$  and  $\omega_{TKS}$ , such that for all  $t \in [t_0, 1]$ ,

$$(4.10) \quad \text{OSC}_M(\varphi_t - \varphi_1) \leq \hat{C}(1 + I_{\omega_{TKS}}(\varphi_t - \varphi_1)).$$

In particular, let  $\varphi' = \varphi_t - \varphi_1$ , then we get that

$$\tilde{F}_{\omega_{TKS},\eta}(\varphi') \geq CI_{\omega_{TKS}}^{\frac{1}{4n+5}}(\varphi') - C',$$

i.e.

$$(4.11) \quad \tilde{F}_{\omega_g,\eta}(\varphi_t) - \tilde{F}_{\omega_g,\eta}(\varphi_1) \geq CI_{\omega_{TKS}}^{\frac{1}{4n+5}}(\varphi') - C'.$$

Further more, we have that

$$\tilde{F}_{\omega_g,\eta}(\varphi_t) - \tilde{F}_{\omega_g,\eta}(\varphi_1) \leq (1-t)((\tilde{I}_{\omega_g}(\varphi_1) - \tilde{J}_{\omega_g}(\varphi_1)) - (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)))$$

By direct calculation, we get that

$$\begin{aligned} & (\tilde{I}_{\omega_g}(\varphi_1) - \tilde{J}_{\omega_g}(\varphi_1)) - (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) \\ &= \frac{1}{V} \int_M \varphi_t (e^{\theta_X(\omega_{\varphi_t})} \omega_{\varphi_t}^n - e^{\theta_X(\omega_{\varphi_1})} \omega_{\varphi_1}^n) + \tilde{J}_{\omega_{\varphi_1}}(\varphi_t - \varphi_1) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{V} \int_M \varphi_t (e^{\theta_X(\omega_{\varphi_t})} \omega_{\varphi_t}^n - e^{\theta_X(\omega_{\varphi_1})} \omega_{\varphi_1}^n) \\ &= \frac{1}{V} \int_0^1 \int_M (\varphi_t - \varphi_1)(\Delta_{s,t} + X)(\varphi_t) e^{\theta_X(s\omega_{\varphi_t} + (1-s)\omega_{\varphi_1})} \omega_{\varphi_1+s(\varphi_t-\varphi_1)}^n \wedge ds \\ &\leq C \text{OSC}_M(\varphi_t - \varphi_1) \end{aligned}$$

with the notation that  $\Delta_{s,t} = \Delta_{\omega_{\varphi_1+s(\varphi_t-\varphi_1)}}$ , for some uniform constant  $C$  depending only on  $X$  and  $\omega_{TKS}$ . So we get that

$$\tilde{F}_{\omega_{TKS},\eta}(\varphi_t - \varphi_1) \leq C''(1-t)(1 + I_{\omega_{TKS}}(\varphi_t - \varphi_1)).$$

Combining with (4.11), we get the inequality as follow with  $\alpha = \frac{4n+4}{4n+5}$ :

$$C \frac{I_{\omega_{TKS}}(\varphi_t - \varphi_1)}{1 + I_{\omega_{TKS}}^\alpha(\varphi_t - \varphi_1)} - C' \leq CI_{\omega_{TKS}}^{1-\alpha}(\varphi_t - \varphi_1) - C' \leq C''(1-t)(1 + I_{\omega_{TKS}}(\varphi_t - \varphi_1)),$$

i.e.

$$(4.12) \quad \frac{I_{\omega_{TKS}}(\varphi_t - \varphi_1)}{1 + I_{\omega_{TKS}}^\alpha(\varphi_t - \varphi_1)} [C - C''(1-t)(1 + I_{\omega_{TKS}}^\alpha(\varphi_t - \varphi_1))] \leq C''(1-t) + C',$$

for some uniform constants  $C$ ,  $C'$  and  $C''$  depending only on  $X$ ,  $\eta$  and the geometry of  $(M, \omega_{TKS})$ . We can suppose that there exists  $t' \in [t_0, 1]$  such that

$$(4.13) \quad C''(1-t')(1 + I_{\omega_{TKS}}^\alpha(\varphi_t - \varphi_1)) = \frac{C}{2},$$

Then (4.12) and (4.13) implies that  $I_{\omega_{TKS}}(\varphi_{t'} - \varphi_1) \leq \tilde{C}$  and  $(1-t') \geq \hat{C}$ , where  $\tilde{C}$  and  $\hat{C}$  are two uniform constants depending only on  $X$ ,  $\eta$  and the geometry of  $(M, \omega_{TKS})$ , and also implies the inequality required by choosing  $t = t'$  and combining with (4.8) and (4.10).

If (4.13) is not true, we must have that

$$C''(1-t_0)(1+I_{\omega_{TKS}}^\alpha(\varphi_{t_0}-\varphi_1))<\frac{C}{2},$$

and it follows that

$$I_{\omega_{TKS}}(\varphi_{t_0}-\varphi_1)<\tilde{C}$$

for some uniform constant  $\tilde{C}$  depending only on  $X$ ,  $\eta$  and the geometry of  $(M, \omega_{TKS})$ . Then choosing  $t = t_0$ , (4.8) and (4.10) implies the inequality required.

**Case 2:** For all  $t$  satisfying  $((1-t)\beta)^{\frac{1}{4(n+1)}} \leq \frac{\sigma}{32C_0}$ , we have

$$(1-t)\|\varphi_t\|_{C^0(M)}((1-t)\beta)^{\frac{1}{4(n+1)}}<\frac{\sigma}{32C_0}$$

where  $C_0$  is the constant in (4.6). With  $1-t_0 = \frac{1}{\beta}(\frac{\sigma}{32C_0})^{4n+4}$ , we have

$$\|\varphi_{t_0}\|_{C^0(M)}<\beta(\frac{\sigma}{32C_0})^{-4n-4}.$$

Similarly to the discussion to getting (4.9), we get that

$$(4.14) \quad \begin{aligned} \text{OSC}_M(\varphi_{t_0}-\varphi_1) &\leq \frac{1}{2} + 12e^\beta(1-t_0)\|\varphi_t\|_{C^0(M)} \\ &\leq C', \end{aligned}$$

where  $C'$  is a uniform constant depending only on  $X$ ,  $\eta$  and  $\omega_{TKS}$ . Then the inequality required is a consequence of (4.8) and (4.14) while  $t = t_0$ .  $\square$

*Remark 4.1.* For the theorem above, we have that

- 1).  $\tilde{\mu}_{\omega_{TKS}, \eta}$  is proper is just an easy consequence of the theorem above and (2.4),
- 2). The inequality in Theorem 1.1 is a consequence of the co-cycle proposition, (2.2) and the theorem above.

As a consequence of Theorem 1.1 and 2.2, we have the following corollary,

**Corollary 4.3.** *The following conditions are equivalent:*

- (1).  $0 < \beta < R(X)$
- (2).  $\exists \eta \in (1-\beta)\mathcal{K}_X^0(\omega_0)$ , there is a twisted Kähler-Ricci soliton in  $\mathcal{K}_X(\omega_0)$  such that

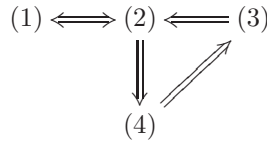
$$\text{Ric}(\omega) = \beta\omega + \eta + L_X\omega,$$

- (3).  $\forall \eta \in (1-\beta)\mathcal{K}_X^0(\omega_0)$ , there exists a twisted Kähler-Ricci soliton such that

$$\text{Ric}(\omega) = \beta\omega + \eta + L_X\omega,$$

- (4).  $\forall \eta \in (1-\beta)\mathcal{K}_X^0(\omega_0)$ , the twisted Mabuchi  $K$ -energy  $\tilde{\mu}_{\omega, \eta}$  is  $\tilde{J}$ -proper, or the Morse-Trudinger inequality is hold.

*Proof.* We will do this as follow:



(2) $\Rightarrow$ (1), and (3) $\Rightarrow$ (2) are obvious.



- (1) $\Rightarrow$ (2) By the definition of  $R(X)$ , there exists  $\hat{\omega} \in \mathcal{K}_X(\omega_0)$ , such that

$$\hat{\eta} = Ric(\hat{\omega}) - \beta\hat{\omega} - L_X\hat{\omega},$$

and it is easy to check that  $\hat{\eta} \in (1 - \beta)\mathcal{K}_X^0(\omega_0)$ .

- (4) $\Rightarrow$ (3) This is an easy consequence of Theorem 1.1.
- (2) $\Rightarrow$ (4) Assuming that

$$\hat{\eta} = Ric(\hat{\omega}) - \beta\hat{\omega} - L_X\hat{\omega}$$

is a  $(1, 1)$ -form in  $(1 - \beta)\mathcal{K}_X^0(\omega_0)$  for  $\hat{\omega} \in \mathcal{K}_X(\omega_0)$ , it is easy to check that

$$\int_M \hat{\eta} \wedge \hat{\omega}^{n-1} = 1 - \beta > 0,$$

which implies that  $\hat{\eta}$  is strictly positive at one point. Theorem 4.2 implies that  $\tilde{\mu}_{\hat{\omega}, \hat{\eta}}$  is proper, and (4) is just a consequence of the proper proposition and Lemma 2.4.

Then the corollary is proved.  $\square$

*Remark 4.2.* Corollary 1.2 is a consequence of the corollary above.

## 5. THE EXISTENCE OF CONICAL KÄHLER-RICCI SOLITON

Before proving Theorem 1.4, we first prove the existence of conical Kähler-Ricci soliton under the properness of the log modified Mabuchi  $K$ -energy or Ding-functional.

In order to make our computations make sense, we will add an assumption to the divisor considered, or more precisely the line bundle of the divisor. Let  $D \in |L|$  be a divisor, and  $L$  is an ample line bundle over  $M$  such that

$$c_1(L) = \lambda c_1(M).$$

$s$  is a section of  $L$  determining  $D$  and  $H$  is a Hermitian metric of  $L$  with curvature  $\lambda\omega_0$ . Furthermore, we assume that  $s$  satisfies that for some constant  $C$ ,

$$(5.1) \quad |X(\log(|s|_H^2))| < C.$$

According to [16], we know that  $\lambda$  is a rational number in  $(0, 1)$ .

**Lemma 5.1.** *If  $|X(\log(|s|_H^2))| < C$ , then,*

$$\text{Im } X(|s|_H^2) = 0.$$

This lemma is an application of a Stokes' formula by C.T.Simpson (cf. Lemma 5.2 of [20]). We give the details of the proof for the readers' convenience.

*Proof.* Let  $\eta = \text{Im } X(\log |s|_H^2) \bar{\partial} \text{Im } X(\log |s|_H^2) \wedge \omega_0^{n-1}$ . Obviously,  $\int_{M \setminus D} |\eta|_{\omega_0}^2 \omega_0^n$  is bounded. Furthermore,  $M \setminus D$  satisfies the conditions in Lemma 5.2 of [20]. Since  $\partial \bar{\partial} \text{Im } X(\log |s|_H^2) \equiv 0$  on  $M \setminus D$ , the Stokes' formula to  $\eta$  implies that

$$(5.2) \quad \int_{M \setminus D} |\bar{\partial} \text{Im } X(\log |s|_H^2)|_{\omega_0}^2 \omega_0^n = \int_{M \setminus D} \sqrt{-1} d\eta = 0.$$

So  $\text{Im } X(\log |s|_H^2)$  is a constant on  $M \setminus D$ , i.e.

$$(5.3) \quad \text{Im } X(|s|_H^2) = C|s|_H^2, \text{ on } M \setminus D.$$

However,  $|s|_H^2 = 0$  on  $D$  and  $|s|_H^2 > 0$  in  $M \setminus D$ , so  $|s|_H^2$  must achieve its maximum in the inner of  $M \setminus D$ . It is easy to see that  $C = 0$ , i.e.

$$\operatorname{Im} X(|s|_H^2) = 0,$$

on  $M$ . □

For  $\nu \in [0, 1]$ , let  $h_\omega$  be the Ricci potential define by

$$\operatorname{Ric}(\omega) = \gamma(\lambda, \nu)\omega + \lambda\nu\omega_0 + \sqrt{-1}\partial\bar{\partial}h_\omega,$$

normalized by  $\int_M e^{h_\omega}\omega^n = \int_M \omega_0^n$  and  $\gamma(\lambda, \nu) = 1 - \lambda\nu$ . For convenience, we define  $H$  by multiplying a constant such that

$$\int_M \frac{e^{h_{\omega_0}}}{|s|_H^{2\nu}} \omega_0^n = \int_M \omega_0^n.$$

We define the log modified Mabuchi  $K$ -energy and Ding-functional on  $\mathcal{H}_X(M, \omega_0)$  as follow:

$$\tilde{\mu}_{\omega_0, \nu D}(\varphi) = \tilde{\mu}_{\omega_0, \nu \lambda \omega_0}(\varphi) + \frac{\nu}{V} \int_M \log(|s|_H^2) (e^{\theta_X(\omega_\varphi)} \omega_\varphi^n - e^{\theta_X(\omega_0)} \omega_0^n),$$

and

$$\tilde{F}_{\omega_0, \nu D}(\varphi) = \hat{F}_{\omega_0}(\varphi) - \frac{1}{\gamma(\lambda, \nu)} \log\left(\frac{1}{V} \int_M \frac{e^{h_0 - \gamma(\lambda, \nu)\varphi}}{|s|_H^{2\nu}} \omega_0^n\right),$$

with the notation that  $h_0 = h_{\omega_0}$ . We can check that both of them are well-defined and satisfy the cocycle property. Similarly to Lemma 2.3, we have the following lemma for  $\tilde{\mu}_{\omega_0, \nu D}$  and  $\tilde{F}_{\omega_0, \nu D}$ :

**Lemma 5.2.**

$$(5.4) \quad \tilde{\mu}_{\omega_0, \nu D} \geq \gamma(\lambda, \nu) \tilde{F}_{\omega_0, \nu D} + \frac{1}{V} \int_M (h_0 - \theta_X - \nu \log |s|_H^2) e^{\theta_X} \omega_0^n.$$

The Poincaré–Lelong formula implies that  $[D] = \lambda\omega_0 + \sqrt{-1}\partial\bar{\partial}\log |s|_H^2$ . The positive  $(1, 1)$ -form  $\eta_\varepsilon = \lambda\omega_0 + \sqrt{-1}\partial\bar{\partial}\log(|s|_H^2 + \varepsilon^2)$  converges to  $[D]$  as a current while  $\varepsilon \rightarrow 0$ . With this notation, we have that,

**Lemma 5.3.** *If  $\tilde{\mu}_{\omega_0, \nu D}$  or  $\tilde{F}_{\omega_0, \nu D}$  is proper on the function space  $\mathcal{H}_X(M, \omega_0)$ , then  $\tilde{\mu}_{\omega_0, \nu \eta_\varepsilon}$  is proper uniformly for all  $\varepsilon \in (0, 1]$ .*

*Proof.* Since (5.4), we just consider when  $\tilde{\mu}_{\omega_0, \nu D}$  is proper. In order to get the properness of  $\tilde{\mu}_{\omega_0, \nu \eta_\varepsilon}$ , we do computation as follow, for all  $\varphi \in \mathcal{H}_X(M, \omega_0)$ :

$$\begin{aligned} & \tilde{\mu}_{\omega_0, \nu \eta_\varepsilon}(\varphi) - \tilde{\mu}_{\omega_0, \nu D}(\varphi) \\ &= \frac{\nu}{V} \int_M \log \frac{|s|_H^2 + \varepsilon^2}{|s|_H^2} (e^{\theta_X(\omega_\varphi)} \omega_\varphi^n - e^{\theta_X(\omega_0)} \omega_0^n) \\ &\geq \frac{\nu}{V} \int_M \log \frac{|s|_H^2}{|s|_H^2 + \varepsilon^2} e^{\theta_X} \omega_0^n \geq -C, \end{aligned}$$

where  $C$  is independent of  $\varphi$ . The uniform properness of  $\tilde{\mu}_{\omega_0, \nu \eta_\varepsilon}$  is a consequence of the inequality. □

Before proving Theorem 1.3, we recall some facts in [11] which will be useful while getting the Laplace estimate. We denote

$$\omega_\varepsilon = \omega_0 + k\sqrt{-1}\partial\bar{\partial}\chi(\varepsilon^2 + |s|_H^2),$$

where

$$\chi(\varepsilon^2 + t) = \frac{1}{1-\nu} \int_0^t \frac{(\varepsilon^2 + r)^{1-\nu} - \varepsilon^{2-2\nu}}{r} dr,$$

and  $k$  is a sufficiently small number such that  $\omega_\varepsilon$  is a Kähler form for each  $\varepsilon \in (0, 1)$ .  $\omega_\varepsilon \rightarrow \omega^*$  in the sense of current globally on  $M$  and in  $C_{\text{loc}}^\infty$  topology outside  $D$ , where

$$\omega^* = \omega_0 + k\sqrt{-1}\partial\bar{\partial}|s|_H^{2-2\nu}$$

is a conical Kähler metric with cone angle  $2\pi(1-\nu)$  along  $D$ . From [11], we know that the function  $\chi(\varepsilon^2 + t)$  is smooth for any  $\varepsilon > 0$ , and there exist constants  $C > 0$  and  $\gamma > 0$  independent of  $\varepsilon$  such that

$$0 \leq \chi(\varepsilon^2 + t) \leq C,$$

provided that  $t$  belongs to a bounded interval and

$$(5.5) \quad \omega_\varepsilon \geq \gamma\omega_0.$$

Now we give the proof of Theorem 1.3 as follow:

*Proof.* In order to solve (1.9), we will follow the method of [17] to get a uniform global  $C^{0,\tau}$ -estimate, Laplace estimate and local  $C^{k,\alpha}$ -estimate ( $k \in \mathbb{Z}^+$ ) for the following equation:

$$(5.6) \quad Ric(\omega_{\phi_{\varepsilon,t}}) = t\gamma(\lambda, \nu)\omega_{\phi_{\varepsilon,t}} + (1-t)\gamma(\lambda, \nu)\omega_0 + \nu\eta_\varepsilon + L_X\omega_{\phi_{\varepsilon,t}},$$

where  $\eta_\varepsilon$  is given above and  $t \in [\frac{1}{2}, 1]$ ,  $\varepsilon \in (0, 1)$ . With  $\omega_{\phi_{\varepsilon,t}} = \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon,t}$  i.e.

$$\phi_{\varepsilon,t} = \varphi_{\varepsilon,t} + k\chi(|s|_H^2 + \varepsilon^2)$$

the scalar version of (5.6) is

$$(5.7) \quad \frac{(\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon,t})^n}{\omega_0^n} = \frac{e^{h_0 - t\gamma(\lambda, \nu)(\varphi_{\varepsilon,t} + k\chi) - \theta_X - X(k\chi + \varphi_{\varepsilon,t})}}{(|s|_H^2 + \varepsilon^2)^\nu}.$$

### Step 1: $C^0$ and Hölder estimate

To get uniform  $C^0$ -estimate for the Kähler potential  $k\chi + \varphi_{\varepsilon,t}$ , we will follow the proof of Proposition 3.6. We just need the uniform up bound of  $\tilde{\mu}_{\omega_0, \nu\eta_\varepsilon}$ , the uniform properness of Mabuchi  $K$ -energy  $\tilde{\mu}_{\omega_0, \nu\eta_\varepsilon}(\phi_{\varepsilon,t})$  which is true according to the lemma above and uniform lower bound of the Green function for the operator

$$\Delta_{\omega_{\phi_{\varepsilon,t}}} + X$$

for any  $t \in [\frac{1}{2}, 1]$  and  $\varepsilon \in (0, 1]$ . The uniform up bound of  $\tilde{\mu}_{\omega_0, \nu\eta_\varepsilon}(\phi_{\varepsilon,t})$  is given by

$$\begin{aligned} \tilde{\mu}_{\omega_0, \nu\eta_\varepsilon}(\phi_{\varepsilon,t}) &\leq \frac{1}{V} \int_M (h_0 - \nu \log(|s|_H^2 + \varepsilon^2) - \theta_X) \omega_0^n \\ &\leq \frac{1}{V} \int_M (h_0 - \nu \log(|s|_H^2 + \varepsilon^2) - \theta_X) \omega_0^n. \end{aligned}$$

According to Lemma 3.4, we just find two uniform constants  $\tilde{\lambda}$  and  $\tilde{k}$ , such that

$$\begin{aligned} Ric(\omega_{\phi_{\varepsilon,t}}) - L_X\omega_{\phi_{\varepsilon,t}} &\geq \tilde{\lambda}\omega_{\phi_{\varepsilon,t}}, \\ \Delta_{\omega_{\phi_{\varepsilon,t}}} \theta_X(\omega_{\phi_{\varepsilon,t}}) &\leq \tilde{k}. \end{aligned}$$

Since (5.6) and  $\eta_\varepsilon$  and  $\omega_0$  are positive, we just choose

$$\tilde{\lambda} = \frac{1}{2}\gamma(\lambda, \nu).$$

Direct computation shows that

$$\begin{aligned}
 (5.8) \quad & \Delta_{\omega_{\phi_{\varepsilon,t}}} \theta_X(\omega_{\phi_{\varepsilon,t}}) \\
 &= -\gamma(\lambda, \nu) \theta_X(\omega_{\phi_{\varepsilon,t}}) - \lambda \nu \theta_X - X(\theta_X(\omega_{\phi_{\varepsilon,t}})) \\
 &\quad - \nu X(\log(|s|_H^2 + \varepsilon^2)) + \gamma(\lambda, \nu)(1-t)X(\phi_{\varepsilon,t}) + C_{\varepsilon,t},
 \end{aligned}$$

where  $C_{\varepsilon,t}$  is a constant. Lemma 3.5 and the equation (5.1) implies that there exists positive constant  $C'$  which is just dependent on  $\omega_0$ ,  $\lambda$ ,  $\nu$  and  $X$ , such that

$$|X(\log(|s|_H^2 + \varepsilon^2))|, |X(\phi_{\varepsilon,t})| \leq C'.$$

Applying the maximal principle to the equation (5.8), we get that

$$C_{\varepsilon,t} \leq \gamma(\lambda, \nu) \operatorname{OSC}_M |\theta_X(\omega_{\phi_{\varepsilon,t}})| + C''$$

where  $C''$  is a uniform constant dependent on  $\lambda$ ,  $\nu$ ,  $C'$  and the  $C^0$ -norm of  $\theta_X$ . Now we rewrite the equation (5.8) as follow

$$\begin{aligned}
 & \Delta_{\omega_{\phi_{\varepsilon,t}}} \theta_X(\omega_{\phi_{\varepsilon,t}}) \\
 &= -\gamma(\lambda, \nu) \theta_X(\omega_{\phi_{\varepsilon,t}}) - \lambda \nu \theta_X - \|X\|_{\omega_{\phi_{\varepsilon,t}}}^2 - \nu X(\log(|s|_H^2 + \varepsilon^2)) \\
 &\quad + \gamma(\lambda, \nu)(1-t)X(\phi_{\varepsilon,t}) + C_{\varepsilon,t} \\
 &\leq 2\gamma(\lambda, \nu) \operatorname{OSC}_M |\theta_X| + 2\gamma(\lambda, \nu)|X(\phi_{\varepsilon,t})| - \lambda \nu \theta_X \\
 &\quad - \nu X(\log(|s|_H^2 + \varepsilon^2)) + \gamma(\lambda, \nu)(1-t)X(\phi_{\varepsilon,t}) + C'' \\
 &\leq C,
 \end{aligned}$$

where  $C$  is a uniform constant independent on  $\varepsilon$  and  $t$ , so we just choose

$$\tilde{k} = C,$$

which implies the uniform  $C^0$ -estimate for the Kähler potential  $\phi_{\varepsilon,t}$ . In particular, the  $L^p$  norm of the right-hand side of (5.7) is uniformly bounded for some fix  $p > 1$ . The uniform global Hölder estimate for  $\phi_{\varepsilon,t}$  is just an easy consequence of Kołodziej's work [14].

### Step 2: uniform Laplace estimate

We want to prove that there exist a uniform positive constant  $A$  such that for any  $\varepsilon \in (0, 1)$ ,  $\omega_{\phi_{\varepsilon}}$  satisfies that

$$(5.9) \quad \frac{1}{A} \omega_{\varepsilon} \leq \omega_{\phi_{\varepsilon}} \leq A \omega_{\varepsilon}$$

with the notation that  $\omega_{\phi_{\varepsilon}} = \omega_{\phi_{\varepsilon,1}}$ . Rewrite the equation (5.7) at  $t = 1$  as follow:

$$\log \frac{(\omega_{\varepsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon})^n}{\omega_{\varepsilon}^n} = h_{\varepsilon} - \gamma(\lambda, \nu)(\varphi_{\varepsilon} + k\chi) - \theta_X - X(k\chi + \varphi_{\varepsilon})$$

where

$$(5.10) \quad h_{\varepsilon} = h_0 - \log \left[ \frac{\omega_{\varepsilon}^n}{\omega_0^n} (|s|_H^2 + \varepsilon^2)^{\nu} \right],$$

and we denote  $F_{\varepsilon} = \log \left[ \frac{\omega_{\varepsilon}^n}{\omega_0^n} (|s|_H^2 + \varepsilon^2)^{\nu} \right]$  which is bounded independent of  $\varepsilon$  according to [11]. We also have the notation as follow

$$\Delta' = \Delta_{\omega_{\phi_{\varepsilon}}}, \Delta = \Delta_{\omega_{\varepsilon}}.$$

Given  $p \in M$ , we choose a locally geodesic holomorphic basis  $\{w_i\}$ , such that

$$g_{i\bar{j}}(p) = \delta_{ij}, \text{ and } g_{\phi_\varepsilon i\bar{j}}(p) = \delta_{ij} + \delta_{ij}\varphi_{\varepsilon i\bar{j}},$$

where  $\varphi_\varepsilon = \varphi_{\varepsilon,1}$ , and  $g_\varepsilon, g_{\phi_\varepsilon}$  are the local representation of  $\omega_\varepsilon$  and  $\omega_{\phi_\varepsilon}$ . Direct computation shows that

$$\begin{aligned} & \Delta' (\log \operatorname{tr}_{\omega_\varepsilon} (\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)) \\ &= \frac{1}{n + \Delta\varphi_\varepsilon} [\Delta h_\varepsilon - \gamma(\lambda, \nu)(k \Delta \chi + \Delta\varphi_\varepsilon) - \Delta\theta_X(\omega_{\phi_\varepsilon}) \\ & \quad + \frac{\varphi_{\varepsilon j\bar{i}}\varphi_{\varepsilon j\bar{i}}}{(1 + \varphi_{\varepsilon i\bar{i}})(1 + \varphi_{\varepsilon j\bar{j}})} + \frac{1}{1 + \varphi_{\varepsilon i\bar{i}}} R_{\omega_\varepsilon i\bar{i}k\bar{k}} - R_{\omega_\varepsilon i\bar{i}k\bar{k}} \\ & \quad + \frac{\varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} R_{\omega_\varepsilon i\bar{i}k\bar{k}}] - \frac{1}{(n + \Delta\varphi_\varepsilon)^2} \frac{\varphi_{\varepsilon i\bar{i}}\varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}}. \end{aligned} \quad (5.11)$$

By [23],

$$\frac{1}{n + \Delta\varphi_\varepsilon} \frac{\varphi_{\varepsilon j\bar{i}}\varphi_{\varepsilon j\bar{i}}}{(1 + \varphi_{\varepsilon i\bar{i}})(1 + \varphi_{\varepsilon j\bar{j}})} - \frac{1}{(n + \Delta\varphi_\varepsilon)^2} \frac{\varphi_{\varepsilon i\bar{i}}\varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} \geq 0. \quad (5.12)$$

On the other hand, we have

$$\begin{aligned} & \frac{1}{1 + \varphi_{\varepsilon i\bar{i}}} R_{\omega_\varepsilon i\bar{i}k\bar{k}} - R_{\omega_\varepsilon i\bar{i}k\bar{k}} + \frac{\varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} R_{\omega_\varepsilon i\bar{i}k\bar{k}} \\ &= R_{\omega_\varepsilon i\bar{i}k\bar{k}} \left( \frac{1}{1 + \varphi_{\varepsilon i\bar{i}}} - 1 + \frac{\varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} \right) \\ &= \frac{1}{2} R_{\omega_\varepsilon i\bar{i}k\bar{k}} \left( \frac{1 + \varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} + \frac{1 + \varphi_{\varepsilon k\bar{k}}}{1 + \varphi_{\varepsilon i\bar{i}}} - 2 \right), \end{aligned} \quad (5.13)$$

and

$$n = \operatorname{tr}_{\omega_\varepsilon} \omega_0 + k \Delta \chi (|s|_H^2 + \varepsilon^2) \geq k \Delta \chi (|s|_H^2 + \varepsilon^2). \quad (5.14)$$

Further more, by direct computation, the following inequality is true

$$\Delta \theta_X(\omega_{\phi_\varepsilon}) \leq (n + \Delta\varphi_\varepsilon) \max\{\sup X_{,j}^j, 0\} + X^j \varphi_{\varepsilon j\bar{i}i}. \quad (5.15)$$

Combining the equation (5.10), (5.11), (5.12), (5.13), (5.14) and (5.15), we get that

$$\begin{aligned} & \Delta' (\log \operatorname{tr}_{\omega_\varepsilon} (\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)) \\ & \geq \frac{\Delta h_0}{n + \Delta\varphi_\varepsilon} - \frac{\Delta F_\varepsilon}{n + \Delta\varphi_\varepsilon} - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j\bar{i}i}}{n + \Delta\varphi_\varepsilon} \\ & \quad + \frac{1}{2(n + \Delta\varphi_\varepsilon)} R_{\omega_\varepsilon i\bar{i}k\bar{k}} \left( \frac{1 + \varphi_{\varepsilon i\bar{i}}}{1 + \varphi_{\varepsilon k\bar{k}}} + \frac{1 + \varphi_{\varepsilon k\bar{k}}}{1 + \varphi_{\varepsilon i\bar{i}}} - 2 \right). \end{aligned} \quad (5.16)$$

There exists a uniform positive constant  $C_1$  such that

$$\sqrt{-1}\partial\bar{\partial}h_0 \geq -C_1\omega_0.$$

Combining with (5.5), we have that

$$-C_1 n \gamma^{-1} \leq \Delta h_0 \leq \gamma^{-1} (nC_1 + \Delta_{\omega_0} h_0). \quad (5.17)$$

We denote  $\Psi_{\varepsilon, \rho} = \tilde{C} \chi_\rho (|s|_H^2 + \varepsilon^2)$ , where

$$\chi_\rho (|s|_H^2 + \varepsilon^2) = \frac{1}{\rho} \int_0^{|s|_H^2} \frac{(\varepsilon^2 + r)^\rho - \varepsilon^{2\rho}}{r} dr.$$

Taking suitable uniform constants  $\tilde{C}$  and  $\rho$ , [11] have proved the following inequality

$$(5.18) \quad \begin{aligned} \Delta' \Psi_{\varepsilon, \rho} \geq & -\frac{1}{2(n + \Delta\varphi_\varepsilon)} R_{\omega_\varepsilon i \bar{i} k \bar{k}} \left( \frac{1 + \varphi_{\varepsilon i \bar{i}}}{1 + \varphi_{\varepsilon k \bar{k}}} + \frac{1 + \varphi_{\varepsilon k \bar{k}}}{1 + \varphi_{\varepsilon i \bar{i}}} - 2 \right) - \frac{\Delta F_\varepsilon}{n + \Delta\varphi_\varepsilon} \\ & - \frac{C_2}{n + \Delta\varphi_\varepsilon} \left( \frac{1 + \varphi_{\varepsilon i \bar{i}}}{1 + \varphi_{\varepsilon k \bar{k}}} + \frac{1 + \varphi_{\varepsilon k \bar{k}}}{1 + \varphi_{\varepsilon i \bar{i}}} \right) - \frac{C_2}{n + \Delta\varphi_\varepsilon} - C_2 \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon, \end{aligned}$$

for some uniform positive constant  $C_2$ . By (5.16), (5.17) and (5.18), we get that

$$(5.19) \quad \begin{aligned} & \Delta' (\log(n + \Delta\varphi_\varepsilon) + \Psi_{\varepsilon, \rho}) \\ & \geq -\frac{n^2 C_3}{n + \Delta\varphi_\varepsilon} - \frac{C_3}{2} \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - \frac{C_3}{4(n + \Delta\varphi_\varepsilon)} \left( \frac{1 + \varphi_{\varepsilon i \bar{i}}}{1 + \varphi_{\varepsilon k \bar{k}}} + \frac{1 + \varphi_{\varepsilon k \bar{k}}}{1 + \varphi_{\varepsilon i \bar{i}}} \right) \\ & \quad - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon} \\ & \geq -\frac{n^2 C_3}{n + \Delta\varphi_\varepsilon} - C_3 \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon} \\ & \geq -2C_3 \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon}, \end{aligned}$$

for some uniform positive constant  $C_3$ , where we use the inequality

$$\operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon \cdot \operatorname{tr}_{\omega_\varepsilon} \omega_{\phi_\varepsilon} \geq n^2$$

in the last inequality. Choosing  $B = 2C_3 + 1$ , we have that

$$\begin{aligned} & \Delta' (\log(n + \Delta\varphi_\varepsilon) + \Psi_{\varepsilon, \rho} - B\varphi_\varepsilon) \\ & \geq -2C_3 \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - B \Delta' \varphi_\varepsilon - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon} \\ & = -2C_3 \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon + B(\operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - n) - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon} \\ & \geq \operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon - C_4 - \gamma(\lambda, \nu) - \max\{\sup X_{,j}^j, 0\} - \frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon}. \end{aligned}$$

We assume that  $p \in M$  is the point that  $\log(n + \Delta\varphi_\varepsilon) + \Psi_{\varepsilon, \rho} - B\varphi_\varepsilon$  achieve its maximal value. By doing  $\frac{\partial}{\partial w^j}$  on the above function, we have that

$$(5.20) \quad \varphi_{\varepsilon j \bar{i} i} = (n + \Delta\varphi_\varepsilon)(B\varphi_{\varepsilon j} - \Psi_{\varepsilon, \rho, j}).$$

An observation is that

$$|X(\varphi_\varepsilon)| \leq |X(\varphi_\varepsilon + k\chi)| + |X(k\chi)| \leq C_5.$$

Further more, we know that there exists positive number  $k'$  independent of  $\varepsilon$  such that  $\omega_0 + k'\sqrt{-1}\partial\bar{\partial}\Psi_{\varepsilon, \rho} \geq 0$ , Lemma 3.5 implies that  $X(\Psi_{\varepsilon, \rho})$  is uniform bounded independent of  $\varepsilon$ . By multiplying  $X^j$  on both sides of (5.20), we get that

$$\frac{X^j \varphi_{\varepsilon j \bar{i} i}}{n + \Delta\varphi_\varepsilon} \leq C_6$$

for some uniform constant  $C_6$ . On the other hand  $X_{,j}^j$  is a uniformly bounded function. The uniform bound of  $\Psi_{\varepsilon, \rho} - B\varphi_\varepsilon$  implies that

$$\operatorname{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon \leq C_7$$

where  $C_7$  is a uniform constant independent of  $\varepsilon$ . According to [11], we know that  $F_\varepsilon$  is uniformly bounded w.r.t  $\varepsilon$ . We have that

$$\begin{aligned} \mathrm{tr}_{\omega_\varepsilon} \omega_{\phi_\varepsilon} &\leq (\mathrm{tr}_{\omega_{\phi_\varepsilon}} \omega_\varepsilon)^{n-1} \frac{\omega_{\phi_\varepsilon}^n}{\omega_\varepsilon} \\ &\leq C_7 e^{h_0 - F_\varepsilon - \gamma(\lambda, \nu) - \theta_X - X(k\chi + \varphi_\varepsilon)} \\ &\leq C_8. \end{aligned}$$

So we prove that there exists a positive uniform constant  $A$  independent of  $\varepsilon$ , such that (5.9) is true.

Similar to [17] and [33], we can get a uniform local upper bound of  $S = |\nabla_{\omega_0} \omega_{\phi_\varepsilon}|_{\omega_{\phi_\varepsilon}}$ , i.e.  $\exists C(r)$  independent of  $\varepsilon$ , such that

$$S|_{B_p(r)} \leq C(r)$$

where  $p \in B_p(r) \subset M \setminus D$ . The local uniform higher order estimates is just an easy consequence of elliptic Schauder estimates, i.e. we get a conical metric with some Hölder function  $\phi$

$$\omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi$$

which is the limit of  $\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\varepsilon$  globally in the sense of current and locally in the  $C^\infty$ -topology.

What remains is to check that  $\omega_\phi$  satisfies that

$$(5.21) \quad Ric(\omega_\phi) = \gamma(\lambda, \nu) \omega_\phi + \nu[D] + L_X \omega_\phi,$$

globally in the sense of current. Indeed, we have

$$\begin{aligned} &\int_M -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_\phi^n |s|_H^{2\nu}}{\omega_0^n} \wedge \zeta \\ &= \int_M -\sqrt{-1} \log \frac{\omega_\phi^n |s|_H^{2\nu}}{\omega_0^n} \partial \bar{\partial} \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \int_M -\sqrt{-1} \log \frac{\omega_{\phi_\varepsilon}^n (|s|_H^2 + \varepsilon^2)^\nu}{\omega_0^n} \partial \bar{\partial} \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \int_M -\sqrt{-1} (h_0 - \gamma(\lambda, \nu) \phi_\varepsilon - \theta_X - X(\phi_\varepsilon)) \partial \bar{\partial} \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \int_M -\sqrt{-1} \partial \bar{\partial} (h_0 - \gamma(\lambda, \nu) \phi_\varepsilon - \theta_X - X(\phi_\varepsilon)) \wedge \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \int_M (\nu \lambda \omega_0 - Ric(\omega_0) + \gamma(\lambda, \nu) \omega_{\phi_\varepsilon} + L_X \omega_{\phi_\varepsilon}) \wedge \zeta \\ &= \lim_{\varepsilon \rightarrow 0} \int_M (\nu \lambda \omega_0 - Ric(\omega_0) + \gamma(\lambda, \nu) \omega_{\phi_\varepsilon}) \wedge \zeta - \lim_{\varepsilon \rightarrow 0} \int_M \omega_{\phi_\varepsilon} \wedge L_X \zeta \\ &= \int_M (\nu \lambda \omega_0 - Ric(\omega_0) + \gamma(\lambda, \nu) \omega_\phi) \wedge \zeta - \int_M \omega_\phi \wedge L_X \zeta \\ &= \int_M (\nu \lambda \omega_0 - Ric(\omega_0) + \gamma(\lambda, \nu) \omega_\phi) \wedge \zeta + \int_M L_X \omega_\phi \wedge \zeta, \end{aligned}$$

i.e.

$$\int_M Ric(\omega_\phi) \wedge \zeta = \int_M (\gamma(\lambda, \nu) \omega_\phi + \nu[D] + L_X \omega_\phi) \wedge \zeta,$$

for any test  $(n-1, n-1)$ -form  $\zeta$ . So we get that  $\omega_\phi$  satisfies the equation (1.9) globally in the sense of current.  $\square$

## 6. SOME EXISTENCE RESULTS OF CONICAL KÄHLER–RICCI SOLITON

In this section we will get some existence results of conical Kähler–Ricci soliton, according to Theorem 1.3, i.e. finding some suitable  $\lambda$  and  $\nu$  such that  $\tilde{\mu}_{\omega_0, \nu D}$  is proper. This is a generalization of [2], [16] and [21]. We begin this section by proving Theorem 1.4.

*Remark 6.1.* For convenience, we will denote the constant  $\tilde{C}$  as the constant  $C_2$  appeared in Proposition 2.1, which is a positive constant less than 1.

**Definition 6.1** ( $\alpha$ -invariant).

$$\alpha([\omega_0], (1 - \beta)D) = \max \left\{ \alpha > 0 \mid \exists 0 < C_\alpha < \infty, \int_M e^{-\alpha(\varphi - \sup \varphi)} \frac{\Omega}{|s|_H^{2-2\beta}} \leq C_\alpha, \forall \varphi \in \mathcal{H}(M, \omega_0) \right\},$$

where  $\Omega$  is a smooth volume form.

According to [2] or [16], we have that

**Lemma 6.1** ([2] or [16]). *If  $D \in |L|$ , and  $c_1(L) = \lambda c_1(M)$ , then for  $\omega_0 \in c_1(M)$ ,*

$$\alpha([\omega_0], (1 - \beta)D) \geq \min\{\lambda\beta, \alpha(\omega_0), \lambda\alpha(L|_D)\} > 0.$$

**Proposition 6.2.** *For any  $\beta$  satisfies that*

$$\max\left\{\frac{1 - \lambda}{1 - \tilde{C}}, 0\right\} < \beta < \min\left\{\frac{\alpha(\omega_0)}{\tilde{C}}, \frac{\lambda\alpha(L_D|D)}{\tilde{C}}, 1\right\},$$

*the functional  $\tilde{\mu}_{\omega_0, \frac{1-\beta}{\lambda}D}$  is proper on the space  $\mathcal{H}_X(M, \omega_0)$ , and there exists a conical Kähler–Ricci soliton solving equation*

$$(6.1) \quad Ric(\omega_\beta) = \beta\omega_\beta + \frac{1 - \beta}{\lambda}[D] + L_X\omega_\beta.$$

*Proof.* For any

$$0 < t < \alpha([\omega_0], \frac{1 - \beta}{\lambda}D),$$

we have

$$\begin{aligned} \log C_t &\geq \log\left(\frac{1}{V} \int_M e^{-t(\varphi - \sup \varphi)} \frac{e^{\theta_X} \omega_0^n}{|s|_H^{\frac{2-2\beta}{\lambda}}}\right) \\ &\geq \frac{1}{V} \int_M \left[-t(\varphi - \sup \varphi) - \log \frac{|s|_H^{\frac{2-2\beta}{\lambda}} e^{\theta_X + X(\varphi)} \omega_\varphi^n}{e^{\theta_X} \omega_0^n}\right] e^{\theta_X + X(\varphi)} \omega_\varphi^n \\ &\geq t\tilde{I}_{\omega_0}(\varphi) - \frac{1}{V} \int_M \log \frac{|s|_H^{\frac{2-2\beta}{\lambda}} e^{\theta_X + X(\varphi)} \omega_\varphi^n}{e^{\theta_X} \omega_0^n} e^{\theta_X + X(\varphi)} \omega_\varphi^n. \end{aligned}$$



By the definition of  $\tilde{\mu}_{\omega_0, D}$ , we have

$$\begin{aligned}
& \tilde{\mu}_{\omega_0, \frac{1-\beta}{\lambda} D}(\varphi) \\
&= -\beta(\tilde{I}_{\omega_0}(\varphi) - \tilde{J}_{\omega_0}(\varphi)) + \frac{1}{V} \int_M (h_0 - \theta_X)(e^{\theta_X} \omega_0^n - e^{\theta_X + X(\varphi)} \omega_\varphi^n) \\
&\quad - \frac{1-\beta}{\lambda V} \int_M \log |s|_H^2 e^{\theta_X} \omega_0^n + \frac{1}{V} \int_M \log \frac{|s|_H^{\frac{2-2\beta}{\lambda}} e^{\theta_X + X(\varphi)} \omega_\varphi^n}{e^{\theta_X} \omega_0^n} e^{\theta_X + X(\varphi)} \omega_\varphi^n \\
&\geq -\beta(\tilde{I}_{\omega_0}(\varphi) - \tilde{J}_{\omega_0}(\varphi)) + t\tilde{I}_{\omega_0}(\varphi) - C_1 \\
&\geq (t - \beta\tilde{C})\tilde{I}_{\omega_0}(\varphi) - C_1.
\end{aligned}$$

We get that if

$$\max\left\{\frac{1-\lambda}{1-\tilde{C}}, 0\right\} < \beta < \min\left\{\frac{\alpha(\omega_0)}{\tilde{C}}, \frac{\lambda\alpha(L_D|D)}{\tilde{C}}, 1\right\},$$

then  $\tilde{\mu}_{\omega_0, \frac{1-\beta}{\lambda} D}$  is proper.

And the second statement is an easy consequence.  $\square$

Next we prove the first part of Theorem 1.4.

*Proof.* Assume that  $\beta_0$  is a fixed positive number given above such that  $\tilde{\mu}_{\omega_0, \frac{1-\beta_0}{\lambda} D}$  is proper. From the definition of  $\tilde{\mu}_{\omega_0, \frac{1-\beta_0}{\lambda} D}$ , we get that for some positive constant  $C_1, C_2$ ,

$$\begin{aligned}
& \frac{1-\beta_0}{\lambda V} \int_M \log |s|_H^2 (e^{\theta_X + X(\varphi)} \omega_\varphi^n - e^{\theta_X} \omega_0^n) \\
&\geq (C_1 + \beta_0)(\tilde{I}_{\omega_0}(\varphi) - C_2 - \tilde{J}_{\omega_0}(\varphi)) - \frac{1}{V} \int_M \log \frac{e^{\theta_X + X(\varphi)} \omega_\varphi^n}{e^{\theta_X} \omega_0^n} e^{\theta_X + X(\varphi)} \omega_\varphi^n.
\end{aligned}$$

For any  $\beta > \beta_0$ , let  $\xi$  be a positive number letter than 1, such that

$$0 < \frac{\beta - \beta_0 - C_1(1-\beta)}{\xi(1-\beta_0)} < \frac{\beta - \beta_0}{1-\beta_0}.$$

Further more, we denote  $\kappa \in (\frac{\beta - \beta_0 - C_1(1-\beta)}{\xi(1-\beta_0)}, \frac{\beta - \beta_0}{1-\beta_0})$ . Then we have that

$$\begin{aligned}
& \tilde{\mu}_{\omega_0, \frac{1-\beta_0}{\lambda} D}(\varphi) - \kappa \tilde{\mu}_{\omega_0, (1-\xi)\omega_0}(\varphi) \\
&\geq \frac{1}{V} \left(1 - \kappa - \frac{1-\beta}{1-\beta_0}\right) \int_M \log \frac{e^{\theta_X + X(\varphi)} \omega_\varphi^n}{e^{\theta_X} \omega_0^n} e^{\theta_X + X(\varphi)} \omega_\varphi^n \\
&\quad + \left[\frac{(C_1 + \beta_0)(1-\beta)}{1-\beta_0} - \beta + \kappa\xi\right] (\tilde{I}_{\omega_0}(\varphi) - \tilde{J}_{\omega_0}(\varphi)) - C_3 \\
&\geq C_4(\tilde{I}_{\omega_0}(\varphi) - \tilde{J}_{\omega_0}(\varphi)) - C_3.
\end{aligned}$$

Further more, according to Corollary 4.3,  $\tilde{\mu}_{\omega_0, (1-\xi)\omega_0}$  is proper, so  $\tilde{\mu}_{\omega_0, (1-\beta)D}$  is proper.  $\square$

While  $R(X) = 1$ , following [7] and [24], we consider the limit behavior of  $\omega_{\phi_\varepsilon}$  under Gromov–Hausdorff distance for  $\beta$  in Theorem 1.4, where  $\omega_{\phi_\varepsilon}$  is solution of

$$Ric(\omega_{\phi_\varepsilon}) = \beta\omega_{\phi_\varepsilon} + \frac{1-\beta}{\lambda}(\lambda\omega_0 + \sqrt{-1}\partial\bar{\partial}\log(|s|_H^2 + \varepsilon^2)) + L_X\omega_{\phi_\varepsilon}.$$

Before proving this, we recall the result of [28], i.e. the extended Cheeger–Colding theory in Bakry–Emery geometry.

**Theorem 6.3** ([28]). *Let  $(M_i, g_i; p_i)$  be a sequence of  $n$ -dimensional Riemannian manifolds which satisfy*

$$\begin{aligned} Ric(g_i) + \text{hess}(f_i) &\geq -(n-1)^2 \Lambda^2 g \\ \text{vol}_{g_i}(B_p(1)) &\geq v > 0, \text{ and } |\nabla f_i|_{g_i} \leq A. \end{aligned}$$

*Then  $(M_i, g_i; p_i)$  converge to a metric space  $(Y; p_\infty)$  in the pointed Gromov–Hausdorff topology.*

In our case,  $f_\varepsilon = \theta_X(\omega_{\phi_\varepsilon})$ , so we first give a uniform estimate for

$$|\nabla f_\varepsilon|_{\omega_{\phi_\varepsilon}} = |X|_{\omega_{\phi_\varepsilon}}.$$

**Lemma 6.4.**  *$|X|_{\omega_{\phi_\varepsilon}}$  is uniformly bounded with respect to  $\varepsilon$ .*

*Proof.* We will denote  $\theta'_X = \theta_X(\omega_{\phi_\varepsilon})$ , and  $\Delta$  is the  $\bar{\partial}$ -Laplace operator associated to  $\omega_{\phi_\varepsilon}$ . As the computation while getting (3.6), we have that

$$\Delta \theta'_X = -\beta \theta'_X - (1-\beta) \theta_X - \frac{1-\beta}{\lambda} X(\log(|s|_H^2 + \varepsilon^2)) - |X|_{\omega_{\phi_\varepsilon}}^2 + C_1,$$

where  $C_1$  is a uniform constant. Following [28], the Bochner implies that

$$\begin{aligned} &(\Delta + X)(|X|_{\omega_{\phi_\varepsilon}}^2) \\ &= |\partial \bar{\partial} \theta'_X|_{\omega_{\phi_\varepsilon}}^2 - \beta |X|_{\omega_{\phi_\varepsilon}}^2 - (1-\beta) |X|_{\omega_0}^2 - \frac{1-\beta}{\lambda} \sqrt{-1} \partial \bar{\partial} \log(|s|_H^2 + \varepsilon^2)(X, \bar{X}). \end{aligned}$$

Since  $|X(\phi_\varepsilon)|$  is uniformly bounded, we get that

$$|\partial \bar{\partial} \theta'_X|_{\omega_{\phi_\varepsilon}}^2 \geq \frac{(\Delta \theta'_X)^2}{n} \geq \frac{(|X|_{\omega_{\phi_\varepsilon}}^2 - C_2)^2}{n}.$$

Since  $X(|s|_H^2) \leq C_3 |s|_H^2$ , we have that

$$\begin{aligned} &\sqrt{-1} \partial \bar{\partial} \log(|s|_H^2 + \varepsilon^2)(X, \bar{X}) \\ &= \left( \frac{\varepsilon^2 \langle ds, ds \rangle}{(|s|_H^2 + \varepsilon^2)^2} - \frac{\lambda |s|_H^2 \omega_0}{|s|_H^2 + \varepsilon^2} \right)(X, \bar{X}) \\ &\leq \frac{\varepsilon^2 \langle ds, s \rangle \wedge \langle s, ds \rangle}{|s|_H^2 (|s|_H^2 + \varepsilon^2)^2}(X, \bar{X}) \\ &= \frac{\varepsilon^2 (X(|s|_H^2))^2}{4 |s|_H^2 (|s|_H^2 + \varepsilon^2)^2} \\ &\leq \frac{C_3^2 \varepsilon^2 |s|_H^4}{4 |s|_H^2 (|s|_H^2 + \varepsilon^2)^2} \\ &\leq C_4. \end{aligned}$$

Combining the inequalities and equalities above,

$$(\Delta + X)(|X|_{\omega_{\phi_\varepsilon}}^2) \geq \frac{(|X|_{\omega_{\phi_\varepsilon}}^2 - C_2)^2}{n} - \beta |X|_{\omega_{\phi_\varepsilon}}^2 - C_5.$$

Applying the maximum principle, we get that  $|X|_{\omega_{\phi_\varepsilon}}$  is uniformly bounded.  $\square$

**Lemma 6.5.** *The diameter of  $(M, \omega_{\phi_\varepsilon})$  is uniformly bounded. Furthermore, the same thing is true for the volume of geodesic ball  $B_p(1)$  with respect to  $\omega_{\phi_\varepsilon}$ .*

*Proof.* By the result of [18], and the condition that

$$\text{Ric}(\omega_{\phi_\varepsilon}) - L_X \omega_{\phi_\varepsilon} \geq \beta \omega_{\phi_\varepsilon},$$

we get that the diameter of  $(M, \omega_{\phi_\varepsilon})$  is bounded by  $\frac{C_1}{\sqrt{\beta}}$ , which is independent of  $\varepsilon$ .

It is an easy consequence of the volume comparison theorem for Bakry–Emery Ricci curvature of [30] that the volume of geodesic ball  $B_p(1)$  with respect to  $\omega_{\phi_\varepsilon}$  is uniformly bounded from above.  $\square$

**Theorem 6.6.** *The smooth Kähler metric  $\omega_{\phi_\varepsilon}$  converge to  $\omega_\beta$  in the Gromov–Hausdorff topology on  $M$  and in the  $C^\infty$  topology outside  $D$ .*

*Proof.* First, we recall the Laplace estimate of  $\omega_\beta$ , i.e. there exists  $C_1, C_2$  such that

$$C_1 \omega_0 \leq \omega_\beta \leq \frac{C_2}{|s|_H^{\frac{2-2\beta}{\lambda}}} \omega_0.$$

It is easy to see that  $\omega_\beta$  defines a metric on  $M$ , and makes it into a compact length space. This can also be considered as the metric completion of the incomplete Riemannian manifold  $(X \setminus D, \omega_\beta)$ . On the other hand, it is not immediately clear that  $(X \setminus D, \omega_\beta)$  is geodesically convex.

By Theorem 6.3, we denote  $(Y, d_Y)$  to be a sequential Gromov–Hausdorff limit of  $(M, \omega_{\phi_\varepsilon})$ . By Lemma 6.5, we know that  $(Y, d_Y)$  is a compact length space. Since  $\omega_{\phi_\varepsilon}$  converges smoothly to  $\omega_\beta$  locally on  $M \setminus D$ , we obtain a smooth dense open subset  $U$  of  $Y$  endowed with a Riemannian metric  $g_\infty$ , and a surjective local isometry

$$F_\infty : (X \setminus D, \omega_\beta) \rightarrow (U, g_\infty) \text{ (as Riemannian manifolds).}$$

For any  $x, y \in U$ , clearly we have  $d_Y(x, y) \leq d_U(x, y)$ , where  $d_U$  is the metric on  $U$  induced by the Riemannian metric  $g_\infty$ . From this it is easy to see that  $F_\infty$  is a Lipschitz map, so  $F_\infty$  extends to a Lipschitz map from the metric completion  $(X, \omega_\beta)$  to  $Y$ , and the image is closed. Since  $U$  is dense, it follows that  $F_\infty$  is surjective. It is clear that  $Y \setminus U$  is contained in  $F_\infty(D)$ . Since  $D$  has zero codimension one Hausdorff measure with respect to  $\omega_\beta$ , we see that  $Y \setminus U$  also has zero codimension one Hausdorff measure. Then by Theorem 3.7 of [6], we know that  $d_U(x, y) = d_Y(x, y)$  for any  $x, y \in U$ , and  $Y$  is the metric completion of  $(U, d_U)$ . It follows that  $F_\infty$  is an isometry.  $\square$

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